Path integration over compact and noncompact rotation groups[a)]

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Applications of group theoretical methods in the path integral formalism of nonrelativistic quantum theory are considered. Analysis of the symmetry of the Lagrangian leads to the expansion of the short time propagator in matrix elements of unitary irreducible representations of the symmetry group. Identification of the coordinates with the group parameters transforms the path integral to integrals over the group manifold. The integration is performed using the orthogonality of the representations. Compact and noncompact rotation groups are considered, where the corresponding path integral is embedded in Euclidean and pseudo-Euclidean spaces, respectively. The unit sphere and unit hyperboloid may either be viewed as the group manifold itself or at least as a group quotient. In the first case Fourier analysis leads to an expansion in group characters. In the second case an expansion in zonal spherical functions is obtained. As examples the groups $SO(n)$, $SU(2)$, $SO(n-1,1)$, and $SU(1,1)$ are explicitly discussed. The path integral on $SO(n+m)$ and $SO(n,m)$ in bispherical coordinates is also treated.

I. INTRODUCTION

In the year 1948 Feynman[1,2] had established the path integration formalism of quantum theory. In field theories the functional integration has been successfully applied in the last two decades. In nonrelativistic quantum theory, however, not much progress in solving exactly particular problems has been made up to 1979. Only quadratic Lagrangians, including a $1/r^2$ potential, could be integrated due to their Gaussian nature. The breakthrough was made in 1979 by Duru and Kleinert,[3] who solved the path integral (in phase space) of the hydrogen atom for the first time. In configuration space this problem has been treated explicitly by Ho and Inomata[4] (a critique of this work was made in the paper by Kleinert[5]). For later calculations see Ref. 6. This success has become possible by employing new techniques such as local time rescaling and dimensional extension. With these tricks the list of exactly soluble problems has increased rapidly. Common to all these is the fact that the dimensional extension has been used for the realization of the dynamical symmetry of the Lagrangian. For example, the dynamical symmetry of the Coulomb[3,4] and dyonium problem[5] has been utilized by the Kustaanheimo–Stiefel transformation being a nonlinear map from $\mathbb{R}^3$ into $\mathbb{R}^4$. Various problems having $SU(2)$ as dynamical symmetry have become solvable by using similar methods. Examples are the Pöschl–Teller,[7,8] Rosen–Morse,[9,10] Hartmann,[11] and Hulthén potentials.[12] For noncompact groups only the $SU(1,1)$ symmetry of the modified Pöschl–Teller potential[13] and the Kepler problem in a uniformly curved space[14] have been realized. Therefore the path integration on symmetry groups, especially on compact and noncompact rotation groups, is of great importance.

In the Schrödinger theory the solution of symmetric problems is usually simplified by choosing proper coordinates, e.g., spherical polar coordinates for spherically symmetric potentials. In the path integral formalism this transition is not that simple since for non-Cartesian coordinates additional quantum corrections of order $O(\hbar^2)$ do appear in the Lagrangian.[15,16] Indeed, the Feynman integral in the usual sliced-time basis[2,17] is only valid in Cartesian coordinates. The aim of the present paper is to derive a general procedure for the path integral treatment on compact and noncompact rotation groups. For this we have to embed the group manifold in Euclidean or pseudo-Euclidean spaces, respectively. We will proceed as follows.

In the next section we start with the definition of our notation. For this we have to recall some properties of transformation groups and their representations. Section III is devoted to the extension of the Feynman ansatz in pseudo-Euclidean space, in order to include the noncompact groups. This makes necessary a modification of the usual regularization scheme.[16] In Sec. IV we introduce generalized polar coordinates and develop two equivalent methods for performing the angular integration. The first one is the character expansion. In lattice gauge theories this technique, called cluster expansion, is used extensively.[18] Actually, the character expansion of Dosch and Müller,[19] where the cluster expansion of a $SU(2)$ Yang–Mills gauge theory on a two-dimensional lattice is done, looks very similar to the expansion formula of Junker and Inomata,[10] where the path integral on the $SU(2)$ manifold is expanded in $SU(2)$ group characters. However, it will be shown that $SU(2)$ and $SU(1,1)$ are the only simple Lie groups where this technique is applicable in ordinary quantum mechanics. In looking for a method having a wider application we develop an expansion in zonal spherical functions. This technique does indeed work on all homogeneous spaces, which may be viewed as a group quotient G/H. In the last part of Sec. IV the connection between both expansions is shown. Finally we discuss an example for both methods for compact and noncompact groups. As compact groups we choose $SO(n)$ and $SU(2)$.

[a)] Dedicated to Hans Joos on the occasion of his 60th birthday.

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For SU(2) we briefly review the path integral treatment in the Pöschl–Teller problem in order to perform the path integral on $SO(n+m)$ in bispherical coordinates using the group chain $SO(n+m)\supset SO(m)\times SO(n)$. As noncompact groups we will take $SO(n-1,1)$ and $SU(1,1)$. An application of the $SU(1,1)$ propagator is made for the modified Pöschl–Teller potential leading to a path integral treatment on $SO(n,m)$ by using $SO(n,m)\supset SO(m)\times SO(n)$. In the Appendix we give the calculation of the Fourier coefficient for the $SU(1,1)$ expansion which has been omitted by us in Ref. 13.

II. TRANSFORMATION GROUPS AND THEIR REPRESENTATIONS

In order to define our notation we repeat some properties of transformation groups and their representations.[20,21] A group G is called a transformation group of a space $\mathscr{M}$, if one may associate with each element $g\in G$ a transformation $\mathbf{x}\rightarrow g\mathbf{x}$ on $\mathscr{M}$. If there exists for any $\mathbf{x},\mathbf{y}\in\mathscr{M}$ an element g such that $g\mathbf{x}=\mathbf{y}$, then G is called a transitive transformation group and $\mathscr{M}$ a homogeneous space.

Let G be a transitive transformation group of $\mathscr{M}$. Furthermore let $\mathscr{L}$ be a linear vector space of functions $f(\mathbf{x})$, $\mathbf{x}\in\mathscr{M}$ such that

$$f(\mathbf{x})\in\mathscr{L}\Leftrightarrow f(g\mathbf{x})\in\mathscr{L}, \tag{2.1}$$

for any $g\in G$. With $f\in\mathscr{L}$ and $g\in G$ a representation of the group G is given by

$$D(g)f(\mathbf{x})=f(g^{-1}\mathbf{x}). \tag{2.2}$$

Choose $\mathscr{L}$ to be the Hilbert space of square integrable functions with respect to a group invariant measure $d\mu(\mathbf{x})$ on $\mathscr{M}$. Then the above representation is unitary relative to the scalar product

$$(f_1,f_2)=\int_{\mathscr{M}} f_1^*(\mathbf{x})\,f_2(\mathbf{x})d\mu(\mathbf{x}). \tag{2.3}$$

Such a representation is called a regular representation. For compact groups the regular representation is decomposable into a direct sum of unitary irreducible representations D^l of this group on $\mathscr{M}$. (A generalization for noncompact groups may be found in Chap. 5 of Ref. 21.) They form a complete basis in the Hilbert space.

Take $D^l(g)$ to be a unitary irreducible representation of G in the Hilbert space $\mathscr{L}$. Furthermore, let H be a subgroup of G which leaves the nonzero vector $a\in\mathscr{L}$ invariant, i.e.,

$$D^l(h)a=a,\quad h\in H\subset G. \tag{2.4}$$

Then $D^l(g)$ is called representation of class 1 relative to H. With each vector $f\in\mathscr{L}$ we may associate a scalar function

$$f^l(g)=(D^l(g)f,a). \tag{2.5}$$

Here $f^l(g)$ is called spherical function of the representation $D^l(g)$. Choosing a basis $\{b_i\}$ in $\mathscr{L}$ such that $b_0=a$, the matrix elements of $D^l(g)$ are given by

$$d^l_{nm}(g)=(D^l(g)b_m,b_n). \tag{2.6}$$

The $d^l_{0m}(g)$ are called associate spherical functions and the $d^l_{00}(g)$ are the zonal spherical functions. Obviously,

$$d^l_{m0}(gh)=d^l_{m0}(g),\quad d^l_{00}(h^{-1}gh)=d^l_{00}(g). \tag{2.7}$$

The spherical functions are eigenfunctions of the Laplace–Beltrami operator on the homogeneous space $\mathscr{M}=G/H$. The Hilbert space is spanned by a complete set $\{l\}$ of associate spherical functions.

Finally we give the general Fourier analysis on compact and noncompact groups[22]:

$$\begin{aligned} f(g)&=\sum_l d_l\sum_{m,n}\hat{f}_{mn}(l)d^l_{nm}(g),\\ \hat{f}_{mn}(l)&=\int_G f(g)d^l_{mn}(g^{-1})dg. \end{aligned}\tag{2.8}$$

The sum Σ_l is to be taken over the complete set $\{l\}$. For compact groups d_l is the dimension of the representation. However, we will call d_l the dimension also in the case of infinite-dimensional unitary representations of noncompact groups. In this case we may take

$$\int_G d^l_{mn}(g)d^{l'*}_{m'n'}(g)dg=\frac{\delta(l,l')}{d_l}\delta_{mm'}\delta_{nn'} \tag{2.9}$$

as a definition for d_l. In (2.9) $\delta(l,l')$ stands for $\delta(l-l')$ in the continuous and for $\delta_{ll'}$ in the discrete case, as noncompact groups in general contain both series. For the continuous series Σ_l is replaced by an integral in (2.8).

III. THE FEYNMAN PROPAGATOR IN PSEUDO-EUCLIDEAN SPACE

According to Feynman[1,2] the nonrelativistic quantum propagator $K(\mathbf{r}_b,\mathbf{r}_a;t_b-t_a)$ is given by a functional integral over the action,

$$\begin{aligned} K(\mathbf{r}_b,\mathbf{r}_a;t_b-t_a)&=\int_{\mathbf{r}_a=\mathbf{r}(t_a)}^{\mathbf{r}_b=\mathbf{r}(t_b)}\exp\left\{\frac{i}{\hbar}S[\mathbf{r}]\right\}\mathscr{D}\mathbf{r}(t)\\ &=\int_{\mathbf{r}_a=\mathbf{r}(t_a)}^{\mathbf{r}_b=\mathbf{r}(t_b)}\exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b}L\,dt\right\}\mathscr{D}\mathbf{r}(t). \end{aligned}\tag{3.1}$$

On the sliced-time basis the path integral in n dimensions is usually written as

$$\begin{aligned} K(\mathbf{r}_b,\mathbf{r}_a;t_b-t_a)=\lim_{N\to\infty}\int\prod_{j=1}^{N}\exp\left\{\frac{i}{\hbar}S_j\right\}\\ \times\prod_{j=1}^{N}\left(\frac{m}{2\pi i\hbar\epsilon}\right)^{n/2}\prod_{j=1}^{N-1}d^n\mathbf{r}_j, \end{aligned}\tag{3.2}$$

with the short time action

$$S_j=(m/2\epsilon)[(\Delta x_j^1)^2+\cdots+(\Delta x_j^n)^2]-V(\mathbf{r}_j)\epsilon. \tag{3.3}$$

For convenience we have chosen an equidistant time slicing $N\epsilon=t_b-t_a$, x^μ ($\mu=1,...,n$) are the Cartesian coordinates of $\mathbf{r}$ and $\Delta x_j^\mu=x_j^\mu-x_{j-1}^\mu$.

In many physically interesting problems the Lagrangian corresponding to (3.3) has a symmetry, which means that it is invariant under group transformations of the symmetry group. Therefore the Hamiltonian of the system may be expressed by Casimir invariants of the dynamical symmetry group. The wave functions correspond to unitary irreducible representations in the Hilbert space. This is the well-known procedure used by the algebraic method.[23]

However, the symmetry of the action may be also very useful in the path integral treatment. Expanding the phase $\exp\{(i/\hbar)S_j\}$ via the Fourier decomposition (2.8) in a series of unitary irreducible representations, the path integral may be performed (at least partially) using the orthogonality (2.9) of the matrix elements. The use of this group property in path integration has already been suggested in 1970 by Dowker.[24] The above path integral (3.2) is defined on a Euclidean space with metric $g_{\mu\nu} = \delta_{\mu\nu}$. As Feynman[1] has already mentioned a generalization to an indefinite metric

$$g_{\mu\nu} = \text{diag}\,\{\underbrace{+1,...,+1}_{p\text{ times}},\underbrace{-1,...,-1}_{q\text{ times}}\} \tag{3.4}$$

is possible. The pseudo-Euclidean space will be denoted by $E_{p,q}$. With metric (3.4) the short time action is given by

$$S_j = (m/2\epsilon)[(\Delta x_j^1)^2 + \cdots + (\Delta x_j^p)^2 - (\Delta x_j^{p+1})^2 - \cdots - (\Delta x_j^{p+q})^2] - V(\mathbf{r}_j)\epsilon. \tag{3.5}$$

In order to match the boundary condition

$$\lim_{t_b\to t_a} K(\mathbf{r}_b,\mathbf{r}_a;t_b - t_a) = \delta(\mathbf{r}_b - \mathbf{r}_a), \tag{3.6}$$

the measure has to be chosen in the following way:

$$K(\mathbf{r}_b,\mathbf{r}_a;t_b - t_a) = \lim_{N\to\infty}\int\prod_{j=1}^{N}\exp\left\{\frac{i}{\hbar}S_j\right\} \times\prod_{j=1}^{N}\left(\frac{m}{2\pi i\hbar\epsilon}\right)^{p/2}\left(\frac{mi}{2\pi\hbar\epsilon}\right)^{q/2}\prod_{j=1}^{N-1} d^{p+q}\mathbf{r}_j. \tag{3.7}$$

For $p=n$ and $q=0$ we recover the Euclidean propagator (3.2). The short time action (3.5) still remains invariant under some group transformation depending on $V(\mathbf{r})$. Therefore the above arguments are valid in the pseudo-Euclidean space, too. Here the symmetry group will be in general noncompact.

However, the above extension of the Feynman ansatz to $E_{p,q}$ requires some modification of the usual path integral formalism. First we have to regularize the path integral in the following way: *Integration over compact coordinates $x^1,...,x^p$ is regularized, as usual, by a mass having a small positive imaginary part, $m\to m+i\eta$ ($\eta>0$), that over the noncompact ($x^{p+1},...,x^{p+q}$), however, by a small negative imaginary part of the mass, $m\to m-i\eta$.*

For $q=0$, i.e., the Euclidean case, this reduces to the prescription of Langguth and Inomata.[16] Second, due to the topology of $E_{p,q}$ the scalar product

$$(\mathbf{r},\mathbf{r}) = (x^1)^2 + \cdots + (x^p)^2 - (x^{p+1})^2 - \cdots - (x^{p+q})^2 \tag{3.8}$$

can be positive, negative, or zero. Consequently, the space $E_{p,q}$ may be divided into three different subspaces T_α:

$$\begin{aligned} T_{+1} &= \{\mathbf{r}|(\mathbf{r},\mathbf{r})>0\}, \quad \text{timelike},\\ T_{-1} &= \{\mathbf{r}|(\mathbf{r},\mathbf{r})<0\}, \quad \text{spacelike},\\ T_0 &= \{\mathbf{r}|(\mathbf{r},\mathbf{r})=0\}, \quad \text{lightlike}. \end{aligned} \tag{3.9}$$

Integration over Cartesian coordinates in $E_{p,q}$ is similar to the usual one in E_n. We still have Gaussian integrals. Up to signs the methods of path integration in Cartesian coordinates on E_n may be applied here similarly.[2,17,25] The propagator has contributions from space-, time- and lightlike paths. And also from paths intersecting different regions T_α.

Systems that may evolve only along one kind of path are also physically interesting. For example, quantum mechanics on a space of constant negative curvature may be discussed in one region T_α of $E_{n-1,1}$ (see Ref. 26).

In this paper we will explore the path integral on such subspaces $T_\alpha \in E_{p,q}$. For this we introduce generalized polar coordinates r and θ^μ, $\mu = 1,...,p+q-1$. In general we have

$$x^\nu = re^\nu(\theta^1,...,\theta^{p+q-1}), \quad \nu = 1,...,p+q. \tag{3.10}$$

The functions e^ν define a unit vector in T_α,

$$\mathbf{e} = (e^1,...,e^{p+q}). \tag{3.11}$$

The set of all such vectors forms a hyperboloid $\mathscr{H}_\alpha \in T_\alpha$. We will call $\mathscr{H}_\alpha$ the unit sphere of T_α,

$$\mathscr{H}_\alpha = \{\mathbf{e}|(\mathbf{e},\mathbf{e}) = \alpha\}, \quad \alpha = 1,-1,0. \tag{3.12}$$

To be more explicit one should also distinguish the nonconnected regions of T_α.

The short time action of the free system on $T_{\pm 1}$ reads in polar coordinates ($0 \leqslant r_j < \infty$, $\Delta r_j = r_j - r_{j-1}$)

$$S_j = \pm(m/2\epsilon)\Delta r_j^2 \pm (m/\epsilon)r_j r_{j-1}[1 \mp (\mathbf{e}_j,\mathbf{e}_{j-1})]. \tag{3.13}$$

In T_0 we have $S_j = -(\mathbf{e}_j,\mathbf{e}_{j-1})mr_jr_{j-1}/\epsilon$. The corresponding path integral separates into an angular and radial part.

$$K(\mathbf{r}_b,\mathbf{r}_a;t_b - t_a) = \lim_{N\to\infty}\int\prod_{j=1}^{N}\exp\left\{\frac{i}{\hbar}S_j\right\}\prod_{j=1}^{N}\left(\frac{m}{2\pi i\hbar\epsilon}\right)^{p/2} \times\left(\frac{mi}{2\pi\hbar\epsilon}\right)^{q/2}\prod_{j=1}^{N-1} r_j^{p+q-1}\,dr_j\,d^{p+q-1}\Omega_j. \tag{3.14}$$

IV. PATH INTEGRATION IN GENERALIZED POLAR COORDINATES

In this section we derive a general procedure for the angular path integration on $E_{p,q}$ using group theoretical methods.

Let G be a transformation group of $\mathscr{H}_\alpha$, i.e.,

$$\mathbf{e} = g\mathbf{a}. \tag{4.1}$$

The $n\times n$ matrix representation $g\in G$ ($n = p+q$) maps the fixed vector $\mathbf{a}$ into the vector $\mathbf{e}$, both being unit vectors on $\mathscr{H}_\alpha$. In (4.1) the vectors $\mathbf{e}$ and $\mathbf{a}$ have to be in the same subspace T_α. The unit sphere $\mathscr{H}_\alpha$ is covered by all possible rotations (4.1). For example, we have

$$\begin{aligned} \mathbf{e}^2 &= +1\colon\ \mathbf{a} = (+1,0,...,0) \quad \text{for } \mathscr{H}_{+1} \text{ with } x^1>0,\\ \mathbf{e}^2 &= +1\colon\ \mathbf{a} = (-1,0,...,0) \quad \text{for } \mathscr{H}_{+1} \text{ with } x^1<0,\\ \mathbf{e}^2 &= -1\colon\ \mathbf{a} = (0,...,0,+1) \quad \text{for } \mathscr{H}_{-1}, \text{ etc.} \end{aligned} \tag{4.2}$$

Note that to $\mathbf{a}\in\mathscr{H}_\alpha$ corresponds $a\in\mathscr{L}$ of Sec. II.

A possible choice of the group G is one that contains $SO(p,q)$, $G \subseteq SO(p,q)$. However, other groups like $SU(u,v)$ may do as well. For example, the unit sphere S^3 in the fourdimensional space E_4 is isomorphic to the group manifold of SU(2). Therefore instead of SO(4) we may choose SU(2) as a transformation group of $S^3 = SO(4)/SO(3)$.

In this paper we restrict ourselves to the cases where $\mathscr{H}_\alpha$ is isomorphic to the group manifold of G, $\mathscr{H}_\alpha \simeq G$, or $\mathscr{H}_\alpha$ is given by a group quotient G/H, $\mathscr{H}_\alpha = G/H$. Here $G = SO(p,q)$ and H is the stationary subgroup of **a**.

A. Expansion in group characters, $\mathscr{H}_\alpha \simeq G$

First we consider the special case $\mathscr{H}_\alpha \simeq G$, where the unit sphere is isomorphic to G. In order to find all rotation groups having this property we use the necessary condition $\dim \mathscr{H}_\alpha = \dim G$. From Table I it follows that SO(2), SO(1,1), SU(2), and SU(1,1) are the only candidates.

For the one-parameter groups SO(2) and SO(1,1) the irreducible representations are one-dimensional [nonunitary for SO(1,1)]. Obviously their characters and zonal spherical functions are identical and therefore these groups will be included in the general theory of the next section. Actually the expansions reduce to the Fourier and Laplace expansions, respectively.

Therefore we are left with the groups SU(2) and SU(1,1). First we consider the group SU(2) which is isomorphic to S^3. The infinitesimal generators are given by Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.3}$$

Defining

$$s^\mu = (i\sigma, \mathbf{1}), \quad \bar{s}^\mu = (-i\sigma, \mathbf{1}), \tag{4.4}$$

the isomorphism between points on the unit sphere $x \in S^3$ corresponding to unit vectors e^μ_x ($e^\mu_x e_{x\mu} = 1$) and the group elements $g \in SU(2)$ may be established by

$$g_x = e^\mu_x s_\mu = \begin{pmatrix} e^4_x + ie^3_x & ie^1_x + e^2_x \\ ie^1_x - e^2_x & e^4_x - ie^3_x \end{pmatrix}, \tag{4.5}$$

$$e^\mu_x = \tfrac{1}{2}\,\mathrm{Tr}(g_x \bar{s}^\mu). \tag{4.6}$$

Note that indeed $\det g_x = 1$, $g^\dagger_x g_x = \mathbf{1}$ and therefore $g \in SU(2)$. From Eq. (4.5) follows

$$\mathrm{Tr}(g_a^{-1} g_b) = 2\mathbf{e}_a \cdot \mathbf{e}_b. \tag{4.7}$$

The explicit identification of the coordinates will be given later.

The group manifold of SU(1,1) is isomorphic to the hyperboloid[13]

$$\begin{aligned} (\mathbf{e},\mathbf{e}) &= e^\mu e_\mu = -(e^1)^2 - (e^2)^2 + (e^3)^2 + (e^4)^2, \\ e^\mu &= (e^1,e^2,e^3,e^4), \quad e_\mu = (-e^1, -e^2, e^3, e^4). \end{aligned} \tag{4.8}$$

The infinitesimal generators may be given by

TABLE I. Solutions for $\dim G = \dim H_\alpha$.

G	$\dim G$	$\dim H_\alpha$	$\dim G = \dim H_\alpha$
$SO(p,q)$	$(p+q)(p+q-1)/2$	$p+q-1$	$p+q=2$
$SU(u,v)$	$(u+v)^2-1$	$2(u+v)-1$	$u+v=2$

$$\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.9}$$

The isomorphism can be established through

$$s_\mu = (i\sigma, \mathbf{1}), \quad \bar{s}_\mu = (-i\sigma^\dagger, \mathbf{1}), \tag{4.10}$$

using the scalar product (4.8) on $E_{2,2}$:

$$g_x = e^\mu_x s_\mu = \begin{pmatrix} e^4_x + ie^3_x & e^1_x - ie^2_x \\ e^1_x + ie^2_x & e^4_x - ie^3_x \end{pmatrix}, \tag{4.11}$$

$$e^\mu_x = \tfrac{1}{2}\,\mathrm{Tr}(g_x \bar{s}^\mu). \tag{4.12}$$

For g_x being an element of SU(1,1) it has to fulfill the following conditions. The pseudounitarity $g^{-1} = \sigma_3 g^\dagger \sigma_3$ is obviously true. But in order to get $\det g = +1$ we must have $(\mathbf{e},\mathbf{e}) = +1$. This means the hyperboloid $\mathscr{H}_{+1}$ has to be chosen. Again we find that the scalar product on $\mathscr{H}_{+1}$ may be written as a trace:

$$\mathrm{Tr}(g_a^{-1} g_b) = 2(\mathbf{e}_a, \mathbf{e}_b). \tag{4.13}$$

We conclude that for the case with $G \simeq \mathscr{H}_\alpha$, the corresponding short time propagator depends only on $\mathrm{Tr}(\hat{g}_j)$, $\hat{g}_j = g_{j-1}^{-1} g_j$ and is therefore invariant under group transformations $f(\hat{g}) \to f(g\hat{g}g^{-1})$. Such functions are called central functions and may be expanded in group characters.[20,21] The Fourier decomposition (2.8) simplifies to

$$\begin{aligned} f(g) &= \sum_l d_l\, \chi^{(l)}(g) \hat{f}(l), \\ \hat{f}(l) &= \frac{1}{d_l} \int_G f(g) \chi^{(l)*}(g) dg. \end{aligned} \tag{4.14}$$

Applying (4.14) to the short time propagator,

$$K(\mathbf{r}_j, \mathbf{r}_{j-1}; \epsilon) = \left(\frac{m}{2\pi i\hbar\epsilon}\right)^{p/2} \left(\frac{mi}{2\pi\hbar\epsilon}\right)^{q/2} \exp\left\{\frac{i}{\hbar} S_j\right\}, \tag{4.15}$$

leads to

$$K(\hat{g}_j; \epsilon) = \sum_l K_l(r_j, r_{j-1}; \epsilon) d_l\, \chi^{(l)}(\hat{g}_j). \tag{4.16}$$

The radial short time propagator $K_l(r_j, r_{j-1}; \epsilon)$ is determined by the Fourier coefficient $\hat{f}(l)$.

Using the group properties

$$\begin{aligned} \int_G \chi^{(l)}(g_{j-1}^{-1} g_j) \chi^{(l')}(g_j^{-1} g_{j+1}) dg_j &= \frac{\delta(l,l')}{d_l} \chi^{(l)}(g_{j-1}^{-1} g_{j+1}), \end{aligned} \tag{4.17}$$

$$\chi^{(l)}(g_a^{-1} g_b) = \sum_{m,n} d^{\,l}_{mn}(g_b) d^{\,l*}_{mn}(g_a),$$

the angular integration can be performed. The $d^{\,l}_{mn}(g)$ are the unitary irreducible representations of G in the Hilbert space $\mathscr{L}$ being infinite dimensional for noncompact groups. Note that $d\Omega$ in Eq. (3.14) is given by $d\Omega = |\mathscr{H}_\alpha| dg$, where $|\mathscr{H}_\alpha|$ denotes the volume of $\mathscr{H}_\alpha$. The resulting propagator reads

$$K(\mathbf{r}_b,\mathbf{r}_a;t_b-t_a) = \sum_{l,m,n} K_l(r_b,r_a;t_b-t_a) Y_{lmn}(\mathbf{e}_b) Y^*_{lmn}(\mathbf{e}_a), \quad (4.18)$$

with

$$Y_{lmn}(\mathbf{e}) = \sqrt{d_l}\, d^l_{mn}(g), \quad (4.19)$$

which we may call generalized harmonics. As for $G=\mathrm{SU}(2)$ they lead to the monopole harmonics of Wu and Yang.[27] Here $K_l(r_b,r_a;t_b-t_a)$ is given by

$$K_l(r_b,r_a;t_b-t_a) = \lim_{N\to\infty} \int \prod_{j=1}^{N} K_l(r_j,r_{j-1};\epsilon) \prod_{j=1}^{N-1} r_j^{p+q-1}\, dr_j. \quad (4.20)$$

The expansions for the compact groups $\mathrm{SO}(2)\simeq\mathrm{U}(1)\simeq S^1$ and $\mathrm{SU}(2)\simeq S^3$ have been discussed in detail by Junker and Inomata.[10] The expansion of the SU(1,1) propagator in $E_{2,2}$ with metric $(+1,+1,-1,-1)$ has been given by the authors.[13] A detailed discussion for SU(1,1) in $E_{2,2}$ with metric (4.8) follows in Sec. VI. The method of character expansion has also been used in the high temperature expansion of field theories on the lattice.[18]

Since as a homogeneous space $\mathscr{H}_\alpha$ usually may be viewed as a quotient G/H, we do need a general scheme for performing the path integration. Such a method may be found by using the expansion of the short time propagator in zonal spherical functions.

B. Expansion in zonal spherical functions, $\mathscr{H}_\alpha = G/H$

In this subsection we consider the case $\mathscr{H}_\alpha = G/H$, where the unit sphere is given by a group quotient. The subgroup $H\subset G$ is the little group of **a**, i.e., $h\mathbf{a}=\mathbf{a}$, $h\in H$.

With (4.1) the scalar product in the short time action (3.13) may be written as

$$(\mathbf{e}_j,\mathbf{e}_{j-1}) = (g_j\mathbf{a},g_{j-1}\mathbf{a}) = (g_{j-1}^{-1} g_j\mathbf{a},\mathbf{a}). \quad (4.21)$$

The short time propagator (4.15) again depends on the group element

$$\hat{g}_j = g_{j-1}^{-1} g_j \quad (4.22)$$

and is invariant with respect to left and right transformations of the subgroup H:

$$K(h\hat{g}_j h^{-1};\epsilon) = K(\hat{g}_j;\epsilon), \quad h\in H. \quad (4.23)$$

Functions having this property may be expanded in zonal spherical functions of the representation of class 1 relative to H (see Ref. 20). The angles θ^μ can be identified with the group parameters of G which do not belong to the subgroup H. As by construction $\mathscr{H}_\alpha = G/H$, $\dim\mathscr{H}_\alpha = \dim G - \dim H$, this identification is always possible.

For functions having the property $f(hgh^{-1}) = f(g)$ the Fourier decomposition (2.8) simplifies to

$$f(g) = \sum_l d_l\, d^l_{00}(g) \hat{f}(l), \qquad \hat{f}(l) = \int_{\mathscr{H}_\alpha} f(g) d^{l*}_{00}(g) d\Gamma. \quad (4.24)$$

In (4.24) the integration over the subgroup H has already been performed using $dg = d\Gamma\, dh$. Here $d\Gamma$ and dh are the normalized measures of $\mathscr{H}_\alpha$ and H, respectively. Note that here $d\Omega$ is given by $|\mathscr{H}_\alpha| d\Gamma$.

Since the short time propagator (4.23) belongs to this class of functions the expansion yields

$$K(\hat{g}_j;\epsilon) = \sum_l K_l(r_j,r_{j-1};\epsilon) d_l\, d^l_{00}(\hat{g}_j), \quad (4.25)$$

where the radial short time propagator $K_l(r_j,r_{j-1};\epsilon)$ is again determined by the Fourier coefficient $\hat{f}(l)$.

Using the group properties

$$d^l_{00}(\hat{g}_j) = \sum_m d^l_{m0}(g_j) d^{l*}_{m0}(g_{j-1}), \qquad \int_{\mathscr{H}_\alpha} d^l_{m0}(g) d^{l'*}_{m'0}(g) d\Gamma = \frac{\delta_{mm'}}{d_l} \delta(l,l'), \quad (4.26)$$

the angular path integration can be performed. The result is

$$K(\mathbf{r}_b,\mathbf{r}_a;t_b-t_a) = \sum_{l,m} K_l(r_b,r_a;t_b-t_a) Y_{lm}(\mathbf{e}_b) Y^*_{lm}(\mathbf{e}_a), \quad (4.27)$$

where

$$Y_{lm}(\mathbf{e}) = \sqrt{d_l}\, d^l_{m0}(g) \quad (4.28)$$

are the hyperspherical harmonics on $\mathscr{H}_\alpha$. Note $K_l(r_b,r_a;t_b-t_a)$ is the remaining radial path integral (4.20).

Since any unit sphere $\mathscr{H}_\alpha$ in $E_{p,q}$ can be viewed as a quotient G/H, the expansion in zonal spherical functions is a general method for performing the path integral on $\mathscr{H}_\alpha$. As examples we will discuss the cases $G=\mathrm{SO}(n)$ and $\mathrm{SO}(n-1,1)$ with $H=\mathrm{SO}(n-1)$.

C. Equivalence of both methods

Above we have discussed two different methods for the path integration on homogeneous spaces. However, as the expansion in zonal spherical functions will always work by construction there arises the question of whether both methods are equivalent or not. In the following we will show that they are indeed identical in the cases where the character expansion does work.

According to Maurin (Ref. 21, p. 237ff), a character of a compact group H can be considered as a zonal spherical function on the group $G=H\times H$. The homogeneous space G/H may be identified with H. For Abelian groups G the characters are also zonal spherical functions of G. Here G need not be compact.

Restricting ourselves to simple Lie groups, the only isomorphism having the above structure is $D_2\simeq A_1\times A_1$ (Ref. 28). The following isomorphisms are obtained[23]:

$$\mathrm{SO}(4)\simeq\mathrm{SU}(2)\times\mathrm{SU}(2)/Z_2, \qquad \mathrm{SO}(2,2)\simeq\mathrm{SU}(1,1)\times\mathrm{SU}(1,1)/Z_2. \quad (4.29)$$

With $\mathrm{SU}(2)/Z_2\simeq\mathrm{SO}(3)$ and $\mathrm{SU}(1,1)/Z_2\simeq\mathrm{SO}(2,1)$ we identify the group manifolds of SU(2) and SU(1,1) with the quotients SO(4)/SO(3) and SO(2,2)/SO(2,1), respectively. Therefore the discussion of SU(2) and SU(1,1) in Sec. IV A contains all simple Lie groups where the expansion in group characters works. By Marinov and Terentyev[29] the fact that the path integral over the SU(n) manifold with

$n>2$ cannot be embedded into a flat space has already been noticed.

V. EXAMPLES FOR COMPACT GROUPS

A. Path integration on SO(n), $\mathscr{H}_\alpha = \mathrm{SO}(n)/\mathrm{SO}(n-1)$

The path integral over $S^{n-1} = \mathrm{SO}(n)/\mathrm{SO}(n-1)$ has been discussed by Marinov and Terentyev[30] for the first time, see also Refs. 10 and 31. However, up to now no explicit path integral treatment has been given.

Introducing spherical polar coordinates

$$\begin{aligned} x^1 &= r\sin\phi^{(n-1)}\cdots\sin\phi^{(1)}, \quad & 0\leqslant r<\infty,\\ x^2 &= r\sin\phi^{(n-1)}\cdots\cos\phi^{(1)}, & 0\leqslant\phi^{(1)}<2\pi,\\ &\vdots & 0\leqslant\phi^{(k)}<\pi \quad (k\neq 1),\\ x^n &= r\cos\phi^{(n-1)}, \end{aligned} \tag{5.1}$$

the Feynman ansatz on E_n reads

$$\begin{aligned} K(\mathbf{r}_b,\mathbf{r}_a;t_b-t_a) &= \lim_{N\to\infty}\int\prod_{j=1}^{N}\exp\left\{\frac{i}{\hbar}S_j\right\}\\ &\quad\times\prod_{j=1}^{N}\left(\frac{m}{2\pi i\hbar\epsilon}\right)^{n/2}\prod_{j=1}^{N-1} r_j^{n-1}\,dr_j\,d^{n-1}\Omega_j, \end{aligned} \tag{5.2}$$

with

$$S_j = (m/2\epsilon)\Delta r_j^2 + (m/\epsilon)r_j r_{j-1}[1-\mathbf{e}_j\cdot\mathbf{e}_{j-1}], \tag{5.3}$$

$$\begin{aligned} d^{n-1}\Omega_j &= \sin^{n-2}\phi_j^{(n-1)}\cdots\sin^2\phi_j^{(3)}\\ &\quad\times\sin\phi_j^{(2)}\,d\phi_j^{(n-1)}\cdots d\phi_j^{(1)}. \end{aligned} \tag{5.4}$$

In order to perform the expansion of the short time propagator in zonal spherical functions we have to recall some properties of the SO(n) representations.[20]

A $n\times n$ matrix representation may be given by a product of rotation matrices

$$\begin{aligned} g &= g^{n-1}\cdots g^k\cdots g^1,\\ g^k &= g_1(\theta_1^k)\cdots g_i(\theta_i^k)\cdots g_k(\theta_k^k), \end{aligned} \tag{5.5}$$

where $g_i(\theta_i^k)$ represents a rotation in the $(i,i+1)$ plane by an angle θ_i^k:

$$\begin{pmatrix} x'^i\\ x'^{i+1}\end{pmatrix} = \begin{pmatrix}\cos\theta_i^k & \sin\theta_i^k\\ -\sin\theta_i^k & \cos\theta_i^k\end{pmatrix}\begin{pmatrix} x^i\\ x^{i+1}\end{pmatrix}. \tag{5.6}$$

The $n(n-1)/2$ parameters θ_i^k are called Eulerian angles of the rotation g,

$$\begin{aligned} &0\leqslant\theta_i^k<\pi, \quad i=2,3,\dots,k,\\ &0\leqslant\theta_1^k<2\pi, \quad k=1,2,\dots,n-1 \end{aligned} \tag{5.7}$$

The associate invariant measure is

$$dg = \prod_{k=1}^{n-1}\left\{\frac{\Gamma((k+1)/2)}{2\pi^{[(k+1)/2]}}\prod_{i=1}^{k}\sin^{i-1}\theta_i^k\,d\theta_i^k\right\}. \tag{5.8}$$

Choosing $\mathbf{a}=(0,\dots,0,1)$ as the stationary vector, each point $\mathbf{e}$ on S^{n-1} may be obtained by a rotation $\mathbf{e}=g\mathbf{a}$. The parameters $\theta_1^{n-1},\dots,\theta_{n-1}^{n-1}$ of g are identical with the polar coordinates $\phi^{(1)},\dots,\phi^{(n-1)}$ of $\mathbf{e}$. The stationary subgroup $H=\mathrm{SO}(n-1)$ of $\mathbf{a}$ is given by the elements $h=g^k$ $(k\neq n-1)$. Integrating (5.8) over all parameters of H yields the normalized volume element on S^{n-1}:

$$d\Gamma = [\Gamma(n/2)/2\pi^{n/2}]\,d\Omega. \tag{5.9}$$

The dimension of the unitary irreducible representation $D^l(g)$ in the Hilbert space is

$$d_l = (2l+n-2)[(l+n-3)!/l!(n-2)!], \quad l=0,1,2,\dots. \tag{5.10}$$

The zonal spherical functions depend only on the parameter θ_{n-1}^{n-1} and are given by Gegenbauer polynomials,

$$d_{00}^{l}(g) = [(n-3)!l!/(l+n-3)!]\times C_l^{(n-2)/2}(\cos\theta_{n-1}^{n-1}). \tag{5.11}$$

Note, that $\Theta\equiv\theta_{n-1}^{n-1}$ is the angle between the vectors $\mathbf{a}$ and $\mathbf{e}$, i.e., $\mathbf{a}\cdot\mathbf{e}=\cos\Theta$.

The associate zonal spherical functions are denoted by $d_{M0}^{l}(g)$, where M stands for the $(n-2)$-tuple

$$\begin{aligned} M &= (m_1,m_2,\dots,m_{n-2}),\\ l &\equiv m_0\geqslant m_1\geqslant\cdots\geqslant m_{n-3}\geqslant|m_{n-2}|. \end{aligned} \tag{5.12}$$

An explicit expression is given by Vilenkin,[20] see also Eq. (5.25).

Now we are well prepared for the expansion of the short time propagator. According to the general theory of Sec. IV, the action may be written as

$$S_j = (m/2\epsilon)\Delta r_j^2 + (m/\epsilon)r_j r_{j-1}[1-\hat{g}_j\mathbf{a}\cdot\mathbf{a}], \tag{5.13}$$

and depends only on the parameter $\Theta\equiv\theta_{n-1}^{n-1}$ of $\hat{g}_j$. Actually we have $\hat{g}_j\mathbf{a}\cdot\mathbf{a}=\cos\Theta$. For the Fourier analysis only the factor $\exp(iz\cos\Theta)$, where $z=-mr_jr_{j-1}/\epsilon\hbar$, has to be considered. We have

$$\exp\{iz\cos\Theta\} = \sum_{l=0}^{\infty} d_l\, d_{00}^{l}(\hat{g}_j)\hat{f}(l), \tag{5.14}$$

$$\hat{f}(l) = \int_{S^{n-1}} e^{iz\cos\Theta} d_{00}^{l*}(\hat{g}_j)\,d\Gamma. \tag{5.15}$$

The integral can be simplified to

$$\begin{aligned} \hat{f}(l) &= \frac{\Gamma(n/2)\Gamma(n-2)l!}{2\sqrt{\pi}\,\Gamma((n-1)/2)\Gamma(n+l-2)}\\ &\quad\times\int_0^{\pi} e^{iz\cos\Theta}C_l^{(n-2)/2}(\cos\Theta)\sin^{n-2}\Theta\,d\Theta \end{aligned} \tag{5.16}$$

and yields (p. 221 in Ref. 32),

$$\hat{f}(l) = \Gamma(n/2)(2/z)^{(n-2)/2} i^l J_{l+(n-2)/2}(z). \tag{5.17}$$

Replacing the Bessel function $J_\nu(z)$ by the modified one $I_\nu(iz)$ leads to the well-known Gegenbauer formula $[\nu=(n-2)/2]$:

$$e^{iz\cos\Theta} = \left(\frac{2}{iz}\right)^{\nu}\Gamma(\nu)\sum_{l=0}^{\infty}(l+\nu)C_l^{\nu}(\cos\Theta)I_{l+\nu}(iz). \tag{5.18}$$

This formula has been used earlier for the path integration in polar coordinates.[10,30,31]

The short time propagator now reads

$$K(\hat{g}_j,\epsilon) = \frac{\Gamma(n/2)}{2\pi^{n/2}}\sum_{l=0}^{\infty} d_l\, d_{00}^{l}(\hat{g}_j)K_l(r_j,r_{j-1};\epsilon). \tag{5.19}$$

The angular path integral can be performed and we find

$$K(\mathbf{r}_b,\mathbf{r}_a;t_b-t_a) = \frac{\Gamma(n/2)}{2\pi^{n/2}} \sum_{l=0}^{\infty} d_l\, d^{\,l}_{00}(g_a^{-1}g_b) K_l(r_a,r_b;t_b-t_a). \tag{5.20}$$

The radial propagator is given by

$$K_l(r_a,r_b;t_b-t_a) = (r_b r_a)^{(1-n)/2} \lim_{N\to\infty} \int \prod_{j=1}^{N} \exp\left\{\frac{i}{\hbar} S^l_j\right\} \times \prod_{j=1}^{N}\left(\frac{m}{2\pi i\hbar\epsilon}\right)^{1/2} \prod_{j=1}^{N-1} dr_j,$$

$$S^l_j = \frac{m}{2\epsilon}\Delta r_j^2 - \left[\left(l+\frac{n-2}{2}\right)^2 - \frac{1}{4}\right]\frac{\hbar^2\epsilon}{2m r_j r_{j-1}}, \tag{5.21}$$

where we have made use of the asymptotic formula[16]

$$I_\nu(iz) = (2\pi i z)^{-1/2} \exp\left\{iz + i\frac{\nu^2-\frac{1}{4}}{2z} + O\left(\frac{1}{z^2}\right)\right\},$$

$$\operatorname{Im} z<0 \Leftrightarrow \operatorname{Im} m>0. \tag{5.22}$$

The propagator (5.20) may be also expressed in terms of hyperspherical harmonics [see Eq. (4.27)]

$$K(\mathbf{r}_b,\mathbf{r}_a;t_b-t_a) = \sum_{l=0}^{\infty} K_l(r_a,r_b;t_b-t_a)\frac{\Gamma(n/2)}{2\pi^{n/2}} \times \sum_M Y_{lM}(\mathbf{e}_b) Y^*_{lM}(\mathbf{e}_a), \tag{5.23}$$

where

$$\sum_M \equiv \sum_{m_1=0}^{m_0}\sum_{m_2=0}^{m_1}\cdots\sum_{m_{n-3}=0}^{m_{n-4}}\sum_{m_{n-2}=-m_{n-3}}^{m_{n-3}}. \tag{5.24}$$

The $Y_{lM}(\mathbf{e})$ are given explicitly by[20]

$$Y_{lM}(\mathbf{e}) = A^l_M \prod_{k=0}^{n-3}\{C^{m_{k+1}+(n-k-2)/2}_{m_k-m_{k+1}}(\cos\phi^{(n-k-1)})\sin^{m_{k+1}}\phi^{(n-k-1)}\}\exp(im_{n-2}\phi^{(1)}),$$

$$(A^l_M)^2 = \frac{1}{\Gamma(n/2)}\prod_{k=0}^{n-3}\left\{\frac{2^{2m_{k+1}+n-k-4}(m_k-m_{k+1})!}{\sqrt{\pi}\,\Gamma(m_{k+1}+m_k+n-k-2)}(n-k-2+2m_k) \times [\Gamma(m_{k+1}+(n-k-2)/2)]^2\right\}. \tag{5.25}$$

They form a complete set on S^{n-1}:

$$\int_{S^{n-1}} Y_{lM}(\mathbf{e}) Y^*_{l'M'}(\mathbf{e})\, d\Gamma = \delta_{ll'}\delta_{MM'}. \tag{5.26}$$

The result (5.23) is identical with that obtained earlier.[10,15,31,33] The SO(n) propagator has already been proved useful in the path integration of the n-dimensional harmonic oscillator and the singular potential $V(\mathbf{r}) = -\alpha/r$ (see Refs. 8 and 31) which is sometimes erroneously called the n-dimensional Coulomb problem.

B. Path integration over the SU(2) manifold, $\mathscr{H}_\alpha = S^3$

The path integration over the SU(2) manifold has recently attracted much attention in the Feynman quantization of various problems having SU(2) as dynamical symmetry. Examples are the nonsymmetric Rosen–Morse,[10] Pöschl–Teller,[7] Hartmann,[11] and Hulthén potentials.[12] Even for the dyonium problem[6] the expansion of the Feynman ansatz in SU(2) matrix elements has been proved useful. In this section we would like to show how this SU(2) expansion, derived by Junker and Inomata,[10] can be incorporated into the general scheme of Sec. IV.

The spinor representation of SU(2) is usually parametrized in Eulerian angles[20]:

$$g(\phi,\theta,\psi) = \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2}\end{pmatrix} \times \begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2\end{pmatrix}\begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2}\end{pmatrix},$$

$$0\leqslant\varphi<2\pi,\quad 0<\theta<\pi,\quad 0\leqslant\psi<4\pi. \tag{5.27}$$

The matrix elements of the $(2J+1)$-dimensional unitary irreducible representation in the Hilbert space are the well-known Wigner functions,

$$d^{\,J}_{mn}(g) = e^{-im\varphi} d^{\,J}_{mn}(\theta) e^{-in\psi},$$

$$J=0,\ \tfrac{1}{2},1,\tfrac{3}{2},\ldots,\qquad -J\leqslant m,n\leqslant J. \tag{5.28}$$

The characters are

$$\chi^{(J)}(g) = \sum_{m=-J}^{J} d^{\,J}_{mm}(g) = \frac{\sin(2J+1)\Theta/2}{\sin\Theta/2}, \tag{5.29}$$

where

$$\cos\left(\frac{\Theta}{2}\right) = \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{(\varphi+\psi)}{2}\right). \tag{5.30}$$

The invariant volume element follows to be

$$dg = (1/16\pi^2)\sin\theta\, d\theta\, d\varphi\, d\psi. \tag{5.31}$$

Comparing (5.27) with the spinor representation (4.5) of Sec. IV suggests the following parametrization of E_4:

$$\begin{aligned} x^1 &= r\sin(\theta/2)\sin((\varphi-\psi)/2), \\ x^2 &= r\sin(\theta/2)\cos((\varphi-\psi)/2), \\ x^3 &= r\cos(\theta/2)\sin((\varphi+\psi)/2), \\ x^4 &= r\cos(\theta/2)\cos((\varphi+\psi)/2), \end{aligned} \qquad \begin{aligned} &0\leqslant\varphi<2\pi, \\ &0\leqslant\theta<\pi, \\ &0\leqslant\psi<4\pi. \end{aligned} \tag{5.32}$$

The corresponding Feynman ansatz reads

$$K(\mathbf{r}_b,\mathbf{r}_a;t_b-t_a) = \lim_{N\to\infty}\int\prod_{j=1}^{N}\exp\left\{\frac{i}{\hbar}S_j\right\} \times \prod_{j=1}^{N}\left(\frac{m}{2\pi i\hbar\epsilon}\right)^2 \prod_{j=1}^{N-1} r_j^3\, dr_j\, 2\pi^2\, dg_j,$$

$$S_j = \frac{m}{2\epsilon}\Delta r_j^2 + \frac{m}{\epsilon} r_j r_{j-1}\left[1-\frac{1}{2}\operatorname{Tr}(\hat{g}_j)\right], \tag{5.33}$$

where we have made use of (4.7).

The expansion of the factor $\exp\{z\,\mathrm{Tr}(g)\}$ in SU(2) characters has already been investigated in lattice gauge theories[19]:

$$\exp\{z\,\mathrm{Tr}(g)\} = \sum_J (2J+1)\frac{1}{z} I_{2J+1}(2z)\chi^{(J)}(g). \tag{5.34}$$

For $2z = mr_j r_{j-1}/i\epsilon\hbar$ we find using the asymptotic formula (5.22) for small ϵ

$$\begin{aligned}\exp&\left\{\frac{imr_j r_{j-1}}{\epsilon\hbar}\left[1-\frac{1}{2}\mathrm{Tr}(\hat{g}_j)\right]\right\}\\ &\approx \frac{1}{2\pi^2}\left(\frac{2\pi i\hbar\epsilon}{mr_j r_{j-1}}\right)^{3/2}\sum_J (2J+1)\\ &\quad\times\exp\left\{-\frac{i}{\hbar}\left[J(J+1)+\frac{3}{16}\right]\frac{2\hbar^2\epsilon}{mr_j r_{j-1}}\right\}\chi^{(J)}(\hat{g}_j).\end{aligned} \tag{5.35}$$

This is the expansion derived by Junker and Inomata.[10] A similar formula has been given by Duru.[7] It contains only integer angular momenta J and therefore does not yield the complete SU(2) propagator.

Performing the angular integration leads to

$$\begin{aligned}K(\mathbf{r}_b,\mathbf{r}_a;t_b-t_a) = \sum_J \frac{2J+1}{2\pi^2} K_J(r_b,r_a;t_b-t_a)\chi^{(J)}(g_a^{-1}g_b),\end{aligned} \tag{5.36}$$

where

$$\begin{aligned}K_J(r_b,r_a;t_b-t_a) &= (r_b r_a)^{-3/2}\lim_{N\to\infty}\int\prod_{j=1}^N \exp\left\{\frac{i}{\hbar}S_j^J\right\}\\ &\quad\times\prod_{j=1}^N\left(\frac{m}{2\pi i\hbar\epsilon}\right)^{1/2}\prod_{j=1}^{N-1}dr_j,\end{aligned} \tag{5.37}$$

$$S_j^J = \frac{m}{2\epsilon}\Delta r_j^2 - \left[J(J+1)+\frac{3}{16}\right]\frac{2\hbar\epsilon}{mr_j r_{j-1}}.$$

The result is identical with (5.20) for $n=4$, as expected.

As already mentioned, the above expansion has been used for various problems having SU(2) symmetry. As an instructive example we may take the one-dimensional Pöschl–Teller potential[34]

$$V(x) = \frac{\hbar^2 a^2}{2m}\left(\frac{\kappa^2-\frac{1}{4}}{\sin^2 ax}+\frac{\lambda^2-\frac{1}{4}}{\cos^2 ax}\right),\quad 0<x<\frac{\pi}{2a}. \tag{5.38}$$

A detailed discussion may be found in Refs. 8, 10, and 35. Here we just state that for $\theta = 2ax$, the Feynman ansatz reads

$$\begin{aligned}K(x_b,x_a;t_b-t_a) &= a\lim_{N\to\infty}\int\prod_{j=1}^N \exp\left\{\frac{i}{\hbar}S_j\right\}\\ &\quad\times\prod_{j=1}^N\left(\frac{m}{2\pi i\hbar a^2\epsilon}\right)^{1/2}\prod_{j=1}^{N-1}\frac{1}{2}d\theta_j,\end{aligned} \tag{5.39}$$

with

$$\begin{aligned}S_j &= \frac{m}{a^2\epsilon}\left(1-\cos\frac{\Delta\theta_j}{2}\right) - \left[\frac{\kappa^2-\frac{1}{4}}{\sin(\theta_j/2)\sin(\theta_{j-1}/2)}\right.\\ &\quad\left.+\frac{\lambda^2-\frac{1}{4}}{\cos(\theta_j/2)\cos(\theta_{j-1}/2)}+\frac{1}{4}\right]\frac{a^2\hbar^2}{2m}\epsilon.\end{aligned} \tag{5.40}$$

For $\kappa,\lambda\in N$ the one-dimensional path integral can be transformed into that of the SU(2) propagator[9,10]:

$$\begin{aligned}K(x_b,x_a;t_b-t_a) &= \frac{a}{4}(\sin\theta_b\sin\theta_a)^{1/2}\exp\left\{-\frac{i\hbar a^2}{8m}(t_b-t_a)\right\}\\ &\quad\times\int_0^{2\pi}\int_0^{4\pi} Q(\theta_b,\varphi_b,\psi_b;\theta_a,0,0;t_b-t_a)\\ &\quad\times\exp\left\{i\left(\frac{\lambda+\kappa}{2}\varphi_b+\frac{\lambda-\kappa}{2}\psi_b\right)\right\}d\psi_b\,d\varphi_b,\end{aligned} \tag{5.41}$$

where $Q(\theta_b,\varphi_b,\psi_b;\theta_a,0,0;t_b-t_a)$ is indeed a path integral over SU(2),

$$\begin{aligned}Q(\theta_b,\varphi_b,\psi_b;\theta_a,0,0;t_b-t_a) &= \lim_{N\to\infty}\int\prod_{j=1}^N\exp\left\{\frac{i}{\hbar}\tilde{S}_j\right\}\prod_{j=1}^N\left(\frac{m}{2\pi i\hbar a^2\epsilon}\right)^{3/2}\\ &\quad\times\prod_{j=1}^{N-1}\frac{1}{8}\sin\theta_j\,d\theta_j\,d\varphi_j\,\psi_j,\end{aligned} \tag{5.42}$$

$$\tilde{S}_j = (m/a^2\epsilon)[1-\tfrac{1}{2}\mathrm{Tr}(\hat{g}_j)]. \tag{5.43}$$

The integration can now be performed and yields

$$\begin{aligned}K(x_b,x_a;t_b-t_a) &= a(\sin\theta_b\sin\theta_a)^{1/2}\sum_{J=(\kappa+\lambda)/2}^{\infty}(2J+1)\\ &\quad\times d^J_{(\lambda+\kappa)/2,(\lambda-\kappa)/2}(\theta_b)\,d^{J*}_{(\lambda+\kappa)/2,(\lambda-\kappa)/2}(\theta_a)\\ &\quad\times\exp\left\{-\frac{i}{\hbar}(2J+1)^2\frac{\hbar^2 a^2}{2m}(t_b-t_a)\right\}.\end{aligned} \tag{5.44}$$

Here J is either an integer or a half-integer depending on $(\lambda+\kappa)/2$. Shifting the summation index yields the standard form [in terms of Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$]

$$K(x_b,x_a;t_b-t_a) = \sum_{n=0}^{\infty}\exp\left\{-\frac{i}{\hbar}E_n(t_b-t_a)\right\}\Psi_n(x_b)\Psi_n^*(x_a), \tag{5.45}$$

where

$$E_n = (2n+\kappa+\lambda+1)^2(\hbar^2 a^2/2m), \tag{5.46}$$

$$\begin{aligned}\Psi_n(x) &= \left[\frac{2a(2n+\kappa+\lambda+1)n!(n+\kappa+\lambda)!}{(n+\kappa)!(n+\lambda)!}\right]^{1/2}\\ &\quad\times\sin^{\kappa+1/2}ax\,\cos^{\lambda+1/2}ax\,P_n^{(\kappa,\lambda)}(1-2\sin^2 ax).\end{aligned} \tag{5.47}$$

Path integration in bispherical coordinates: The above solution of the Pöschl–Teller problem now enables us to perform the path integral in bispherical coordinates

$$
\begin{aligned}
&x^1 = r\sin(\theta/2)\sin\alpha^{(n-1)}\cdots\sin\alpha^{(1)},\\
&\quad\vdots\\
&x^n = r\sin(\theta/2)\cos\alpha^{(n-1)}, && 0\leqslant r<\infty,\\
&x^{n+1} = r\cos(\theta/2)\sin\beta^{(m-1)}\cdots\sin\beta^{(1)}, && 0\leqslant\alpha^{(1)},\beta^{(1)}<2\pi,\\
&\quad\vdots && 0\leqslant\alpha^{(i)},\beta^{(i)},\theta<\pi(i\neq 1).\\
&x^{n+m} = r\cos(\theta/2)\cos\beta^{(m-1)},
\end{aligned}
\tag{5.48}
$$

The Jacobian is

$$
d^{m+n}\mathbf{r} = r^{m+n-1}\,dr\,\frac{1}{2}\sin^{n-1}\frac{\theta}{2}\cos^{m-1}\frac{\theta}{2}\,d\theta\,d^{n-1}\Omega(\alpha)d^{m-1}\Omega(\beta),
\tag{5.49}
$$

where $d\Omega$ is given similar to Eq. (5.4).

The propagator of the free system in E_{m+n} is

$$
K(\mathbf{r}_b,\mathbf{r}_a;t_b-t_a) = \lim_{N\to\infty}\int\prod_{j=1}^{N}\exp\left\{\frac{i}{\hbar}S_j\right\}\prod_{j=1}^{N}\left(\frac{m}{2\pi i\hbar\epsilon}\right)^{(m+n)/2}\prod_{j=1}^{N-1}d^{m+n}\mathbf{r}_j,
\tag{5.50}
$$

$$
\begin{aligned}
S_j = {}&\frac{m}{2\epsilon}\Delta r_j^2+\frac{m}{\epsilon}r_jr_{j-1}\left(1-\cos\frac{\Delta\theta_j}{2}\right)\\
&+\frac{m}{\epsilon}r_jr_{j-1}\sin\frac{\theta_j}{2}\sin\frac{\theta_{j-1}}{2}(1-\mathbf{e}^{\alpha}_{j-1}\cdot\mathbf{e}^{\alpha}_j)+\frac{m}{\epsilon}r_jr_{j-1}\cos\frac{\theta_j}{2}\cos\frac{\theta_{j-1}}{2}(1-\mathbf{e}^{\beta}_{j-1}\cdot\mathbf{e}^{\beta}_j),
\end{aligned}
\tag{5.51}
$$

where $\mathbf{e}^{\alpha}$ and $\mathbf{e}^{\beta}$ are the unit vectors in the subspaces E_n and E_m, respectively.

Guided by the group chain $SO(m+n)\supset SO(m)\times SO(n)$ the integration over the angles $\alpha^{(i)}$ and $\beta^{(i)}$ can be performed analogously to Sec. V A. We find

$$
K(\mathbf{r}_b,\mathbf{r}_a;t_b-t_a) = \sum_{l,\lambda=0}^{\infty}K_{l,\lambda}(r_b,\theta_b;r_a,\theta_a;t_b-t_a)\frac{\Gamma(n/2)}{2\pi^{n/2}}\sum_N Y_{lN}(\mathbf{e}^{\alpha}_b)Y^*_{lN}(\mathbf{e}^{\alpha}_a)\frac{\Gamma(m/2)}{2\pi^{m/2}}\sum_M Y_{\lambda M}(\mathbf{e}^{\beta}_b)Y^*_{\lambda M}(\mathbf{e}^{\beta}_a),
\tag{5.52}
$$

where

$$
\begin{aligned}
K_{l,\lambda}(r_b,\theta_b;r_a,\theta_a;t_b-t_a) = {}&\left(r_br_a\sin\frac{\theta_b}{2}\sin\frac{\theta_a}{2}\right)^{(1-n)/2}\left(r_br_a\cos\frac{\theta_b}{2}\cos\frac{\theta_a}{2}\right)^{(1-m)/2}\\
&\times\lim_{N\to\infty}\int\prod_{j=1}^{N}\exp\left\{\frac{i}{\hbar}\tilde{S}_j\right\}\prod_{j=1}^{N}\left(\frac{m}{2\pi i\hbar\epsilon}\right)\prod_{j=1}^{N-1}r_j\,dr_j\,\frac{1}{2}\,d\theta_j,
\end{aligned}
\tag{5.53}
$$

$$
\tilde{S}_j = \frac{m}{2\epsilon}\Delta r_j^2+\frac{m}{\epsilon}r_jr_{j-1}\left(1-\cos\frac{\Delta\theta_j}{2}\right)-\left[\frac{\mu^2-\frac14}{\sin(\theta_j/2)\sin(\theta_{j-1}/2)}+\frac{\nu^2-\frac14}{\cos(\theta_j/2)\cos(\theta_{j-1}/2)}\right]\frac{\hbar^2\epsilon}{2mr_jr_{j-1}},
\tag{5.54}
$$

with $\mu=l+(n-2)/2$ and $\nu=\lambda+(m-2)/2$. The θ integral is now formal identical with the Pöschl–Teller problem leading to

$$
\begin{aligned}
&K_{l,\lambda}(r_b,\theta_b;r_a,\theta_a;t_b-t_a)\\
&\quad=2\left(r_br_a\sin\frac{\theta_b}{2}\sin\frac{\theta_a}{2}\right)^{(2-n)/2}\\
&\quad\times\left(r_br_a\cos\frac{\theta_b}{2}\cos\frac{\theta_a}{2}\right)^{(2-m)/2}\\
&\quad\times\sum_{J=(\mu+\nu)/2}(2J+1)\\
&\quad\times d^{J}_{(\mu+\nu)/2,(\nu-\mu)/2}(\theta_b)d^{J*}_{(\mu+\nu)/2,(\nu-\mu)/2}(\theta_b)\\
&\quad\times K_J(r_b,r_a;t_b-t_a),
\end{aligned}
\tag{5.55}
$$

where $K_J(r_b,r_a;t_b-t_a)$ is given by Eq. (5.37). Note that (5.53) can only be transformed into a SU(2) integral for even m and n.

VI. EXAMPLES FOR NONCOMPACT GROUPS

Up to now we have dealt only with compact groups, where the final results were already known by other methods. However, the general theory of Secs. III and IV was formulated in such a way that noncompact groups can also be treated. Here we will choose as examples the n-dimensional Lorentz group $SO(n-1,1)$ and $SU(1,1)$. These noncompact groups are often used for scattering problems in quantum theory. Both can be viewed as analytical continuations of $SO(n)$ and $SU(2)$, respectively. Therefore we will keep close to the calculation of the previous section.

A. Path integration on $G=\mathrm{SO}(n-1,1)$, $\mathscr{H}_\alpha \subset \mathrm{SO}(n-1,1)/\mathrm{SO}(n)$

As already mentioned, the space $E_{n-1,1}$ having the metric

$$(\mathbf{r},\mathbf{r}) = (x^1)^2 + \cdots + (x^{n-1})^2 - (x^n)^2 \tag{6.1}$$

consists of topologically different subspaces. Here we will perform the path integral on $\mathscr{H}_{-1} \in T_{-1} = \{\mathbf{r} | (\mathbf{r},\mathbf{r}) < 0\}$, the spacelike subspace. The polar coordinates on T_{-1} may be introduced via

$$\begin{aligned} x^1 &= r \sinh \phi^{(n-1)} \sin \phi^{(n-2)} \cdots \sin \phi^{(1)}, \\ x^2 &= r \sinh \phi^{(n-1)} \sin \phi^{(n-2)} \cdots \cos \phi^{(1)}, \\ &\vdots \\ x^n &= r \cosh \phi^{(n-1)}, \end{aligned} \qquad \begin{aligned} &0 \leqslant r, \quad \phi^{(n-1)} < \infty, \\ &0 \leqslant \phi^{(1)} < 2\pi, \\ &0 \leqslant \phi^{(k)} < \pi \quad (k \neq 1, n-1). \end{aligned} \tag{6.2}$$

The Feynman ansatz reads

$$K(\mathbf{r}_b,\mathbf{r}_a;t_b - t_a) = \lim_{N\to\infty} \int \prod_{j=1}^{N} \exp\left\{\frac{i}{\hbar} S_j\right\} \times \prod_{j=1}^{N} \left(\frac{m}{2\pi i \hbar \epsilon}\right)^{(n-1)/2} \times \left(\frac{mi}{2\pi\hbar\epsilon}\right)^{1/2} \prod_{j=1}^{N-1} d^n \mathbf{r}_j, \tag{6.3}$$

with

$$S_j = -(m/2\epsilon)\Delta r_j^2 - (m/\epsilon) r_j r_{j-1} [1 + (\mathbf{e}_j,\mathbf{e}_{j-1})], \tag{6.4}$$

$$d^{n-1}\Omega = \sinh^{n-2} \phi^{(n-1)} \sin^{n-3} \phi^{(n-2)} \cdots \sin \phi^{(2)} \times d\phi^{(n-1)} \cdots d\phi^{(1)}. \tag{6.5}$$

Before proceeding we have to recall some properties of the $\mathrm{SO}(n-1,1)$ representations.[20]

The $n \times n$ matrix representation may be given by products of hyperbolic and ordinary rotations:

$$g = g^{(n-1)} h, \tag{6.6}$$

where h is a $n \times n$ representation of the maximal compact subgroup $H = \mathrm{SO}(n-1)$ given by Eq. (5.5), and

$$g^{(n-1)} = g_1(\theta_1^{n-1}) \cdots g_k(\theta_k^{n-1}) \cdots g_{n-1}(\theta_{n-1}^{n-1}), \tag{6.7}$$

where $g_k(\theta_k^{n-1})$ $(k \neq n-1)$, is a rotation in the $(k,k+1)$ plane [see Eq. (5.6)]. Here $g_{n-1}(\theta_{n-1}^{n-1})$ is the Lorentz transformation

$$\begin{pmatrix} x'^{n-1} \\ x'^n \end{pmatrix} = \begin{pmatrix} \cosh \theta_{n-1}^{n-1} & \sinh \theta_{n-1}^{n-1} \\ \sinh \theta_{n-1}^{n-1} & \cosh \theta_{n-1}^{n-1} \end{pmatrix} \begin{pmatrix} x^{n-1} \\ x^n \end{pmatrix}. \tag{6.8}$$

The parameter θ_{n-1}^{n-1} is in the interval $0 \leqslant \theta_{n-1}^{n-1} < \infty$ and all others are limited analogously to Eq. (5.7). The invariant volume element may be obtained by analytical continuation of (5.8):

$$dg = \frac{\Gamma(n/2)}{2\pi^{n/2}} \sinh^{n-2} \theta_{n-1}^{n-1} \sin^{n-3} \theta_{n-2}^{n-1} \cdots \sin \theta_2^{n-1} \times d\theta_{n-1}^{n-1} \cdots d\theta_1^{n-1}\, dh, \tag{6.9}$$

dh is the corresponding measure of $h \in \mathrm{SO}(n-1)$.

Taking the northpole $\mathbf{a} = (0,\ldots,0,+1)$ as stationary vector, each $\mathbf{e}$ on the spacelike hyperboloid $\mathscr{H}_{-1}$ may be obtained by the transformation $\mathbf{e} = g\mathbf{a}$. The polar coordinates $\phi^{(1)},\ldots,\phi^{(n-1)}$ of $\mathbf{e}$ are given by the parameters $\theta_1^{n-1},\ldots,\theta_{n-1}^{n-1}$ of $g^{(n-1)}$. The little group is $\mathrm{SO}(n-1)$.

Here $\mathscr{H}_{-1}$ is also called the $(n-1)$-dimensional Lobachevsky space, denoted by Λ^{n-1} (see Ref. 20). Group theoretically we have

$$\Lambda^{n-1} \subset \mathrm{SO}(n-1,1)/\mathrm{SO}(n-1), \tag{6.10}$$

where Λ^{n-1} is a model of a space of constant negative curvature, similar to the way S^{n-1} represents a space of constant positive curvature. Quantum mechanics on spaces with negative curvature is of interest.[26] For example quantum chaos is recently studied on such topologies.[36]

The normalized volume element on $\mathscr{H}_{-1}$, in the sense of $\int_{\mathscr{H}_{-1}} f(\mathbf{r})\delta(\mathbf{r}) d\Gamma = f(0)$, is

$$d\Gamma = [\Gamma(n/2)/2\pi^{n/2}] d^{n-1}\Omega. \tag{6.11}$$

The unitary irreducible representations D^l in the Hilbert space are continuous,[20]

$$\begin{aligned} &\text{fundamental series: } l = -(n-2)/2 + i\rho, \\ &\qquad\qquad -\infty < \rho < +\infty, \\ &\text{complementary series: } -n+2 < l < 0. \end{aligned} \tag{6.12}$$

The zonal spherical functions depend only on the parameter $\Theta \equiv \theta_{n-1}^{n-1}$, $(\mathbf{e},\mathbf{a}) = -\cosh \Theta$ (see Ref. 20):

$$d_{00}^l(g) = 2^{(n-3)/2} \frac{\Gamma((n-1)/2)}{\sinh^{(n-3)/2} \Theta} P_{l+(n-3)/2}^{(3-n)/2}(\cosh \Theta). \tag{6.13}$$

Expressing the Legendre function $P_l^m(z)$ in terms of Gegenbauer functions shows the analytical continuation of (5.11) explicitly:

$$d_{00}^l(g) = \frac{(n-3)!\Gamma(l+1)}{\Gamma(l+n-2)} C_l^{(n-2)/2}(\cosh \Theta) \tag{6.14}$$

The associate spherical functions $d_{K0}^l(g)$ may be written as a product due to Eq. (6.6):

$$d_{K0}^l = d_{K'0}^l(g^{(n-1)}) d_{M0}^k(h), \tag{6.15}$$

with

$$\begin{aligned} K &= (k,m_1,\ldots,m_{n-3}), \\ M &= (m_1,\ldots,m_{n-3}), \\ K' &= (k,0,\ldots,0), \end{aligned} \qquad \begin{aligned} &k \equiv m_0 \geqslant \cdots \geqslant |m_{n-3}|, \\ &k = 0,1,2,\ldots. \end{aligned} \tag{6.16}$$

Actually, $d_{M0}^k(h)$ is the associate spherical function of the subgroup $\mathrm{SO}(n-1)$ [see Eq. (5.25)]. Here $d_{K'0}^l(g^{(n-1)})$ again depends only on Θ and is given by[20]

$$d^{l}_{K'0}(g^{(n-1)})$$

$$= (-1)^k 2^{(n-5)/2}\frac{\Gamma((n-3)/2)\Gamma(l+1)}{\Gamma(n-3)\Gamma(l-k+1)}$$

$$\times\left[(n+2k-3)\frac{\Gamma(n-2)\Gamma(n+k-3)}{k!}\right]^{1/2}$$

$$\times \sinh^{(3-n)/2}\Theta\, P^{(3-n)/2+k}_{l+(n-3)/2}(\cosh\Theta). \qquad (6.17)$$

For the expansion in zonal spherical functions only the fundamental series in (6.12) has to be considered. Vilenkin[20] distinguishes between even and odd dimension [$\mathbf{e} = g\mathbf{a}, f(\mathbf{e})$ being invariant under $SO(n-1)$ transformations]:

$n = 2m + 2$:

$$f(\mathbf{e}) = [(-1)^m 2^{2m+1}\pi^{m+1/2}\Gamma(m+1/2)]^{-1}$$

$$\times\int_{-\infty}^{+\infty}\frac{\Gamma(l+2m)}{\Gamma(l)}\hat{f}(l)d^{l}_{00}(g)d\rho,$$

$n = 2m + 1$:

$$f(\mathbf{e}) = [(-1)^{m+1}2^{2m}\pi^m\Gamma(m)]^{-1}$$

$$\times\int_{-\infty}^{+\infty}\frac{\Gamma(l+2m-1)}{\Gamma(l)}$$

$$\times\cot(\pi l)\hat{f}(l)d^{l}_{00}(g)d\rho, \qquad (6.18)$$

with $l = -(n-2)/2 + i\rho$ and

$$\hat{f}(l) = \int_{\mathscr{H}_a} f(\mathbf{e})d^{l}_{00}(g^{-1})d^{n-1}\Omega. \qquad (6.19)$$

Using some group properties one finds the shorter formulation

$$f(g) = \int_{-\infty}^{+\infty}\frac{|\Gamma((n-2)/2+i\rho)|^2}{|\Gamma(i\rho)|^2\Gamma(n-1)}$$

$$\times\left(1 \pm i\frac{n-2}{2}\rho\right)\hat{f}(l)d^{l}_{00}(g)d\rho, \qquad (6.20)$$

$$\hat{f}(l) = \int_{SO(n-1,1)} f(g)d^{l}_{00}(g^{-1})dg, \qquad (6.21)$$

where the upper sign has to be taken for even dimensions and the lower one for odd n, respectively. However, as the Legendre function $P^{\alpha}_{-1/2+i\rho}(z)$ is symmetric in the index ρ (see Ref. 36), i.e., $P^{\alpha}_{-1/2+i\rho}(z) = P^{\alpha}_{-1/2-i\rho}(z)$, the separation between even and odd n is obsolete. The integration in (6.20) is reducible to one along the positive ρ axis and the substitution

$$\tilde{f}(\cosh\Theta) = 2^{(n-3)/2}\frac{\Gamma(n/2)}{\sqrt{\pi}}\sinh^{(n-3)/2}\Theta f(g) \qquad (6.22)$$

leads to the generalized Mehler transformation[37]:

$$\tilde{f}(t) = \frac{|\Gamma((n-2)/2+i\rho)|^2}{|\Gamma(i\rho)|^2}\int_0^\infty c(\rho)P^{(3-n)/2}_{-1/2+i\rho}(t)d\rho, \qquad (6.23)$$

$$c(\rho) = \int_1^\infty \tilde{f}(t)P^{(3-n)/2}_{-1/2+i\rho}(t)dt. \qquad (6.24)$$

Here n may be an arbitrary complex number.

In the following we consider the Fourier analysis on $SO(n-1,1)$ in the reduced form

$$f(g) = \int_0^\infty 2\frac{|\Gamma((n-2)/2+i\rho)|^2}{|\Gamma(i\rho)|^2\Gamma(n-1)}\hat{f}(l)d^{l}_{00}(g)d\rho, \qquad (6.25)$$

with $\hat{f}(l)$ given by (6.21). Comparison with Eq. (2.8) leads to the definition of the dimension

$$d_l = 2[|\Gamma((n-2)/2+i\rho)|^2/|\Gamma(i\rho)|^2\Gamma(n-1)]. \qquad (6.26)$$

Indeed analytical continuation of the dimension $d^{SO(n)}_l$ for $SO(n)$ in $l \to -(n-2)/2 + i\rho$ gives

$$d^{SO(n)}_l \to d_l(-1)^{(n-2)/2}\begin{cases}1, & \text{for even } n,\\ \coth\pi\rho, & \text{for odd } n.\end{cases} \qquad (6.27)$$

The above definition for d_l is confirmed by the orthogonality

$$\int_{SO(n-1,1)} d^{l}_{K0}(g)d^{l'*}_{K\bar{0}}(g)dg = \frac{\delta(\rho-\rho')}{d_l}\delta_{KL}. \qquad (6.28)$$

For the expansion of the short time propagator we rewrite the action (6.4) using $(\mathbf{e}_j,\mathbf{e}_{j-1}) = (\hat{g}_j\mathbf{a},\mathbf{a}) = -\cosh\Theta$. Note, that now Θ is the parameter θ^{n-1}_{n-1} of the group element $\hat{g}_j = g_{j-1}^{-1}g_j$. Again only the factor $\exp\{z\cosh\Theta\}$ with $z = imr_jr_{j-1}/\hbar\epsilon$ has to be considered. For the Fourier coefficient we have ($\nu = (3-n)/2$)

$$\hat{f}(l) = \frac{\Gamma(n/2)}{2^\nu\sqrt{\pi}}\int_1^\infty e^{zt}(t^2-1)^{-\nu/2}P^{\nu}_{-1/2+i\rho}(t)dt. \qquad (6.29)$$

With $\mathrm{Re}\,\nu < 1 (\Rightarrow n > 1)$ and $\mathrm{Re}\,z < 0$ ($\Rightarrow \mathrm{Im}\,m > 0$) the integral can be performed (p. 194 in Ref. 32):

$$\hat{f}(l) = \frac{\Gamma(n/2)}{\sqrt{\pi}}2^{(n-2)/2}(-z)^{\nu-1/2}K_{i\rho}(-z), \qquad (6.30)$$

where $K_{i\rho}(-z)$ is the modified Bessel function of the third kind. Using the asymptotic form for $|z| \to \infty$ and $|\arg z| < 3\pi/2$,

$$K_{i\rho}(-z) = \sqrt{\frac{\pi}{-2z}}\exp\left\{z + \frac{\rho^2+1/4}{2z} + O\left(\frac{1}{z^2}\right)\right\}, \qquad (6.31)$$

the path integration results in

$$K(\mathbf{r}_b,\mathbf{r}_a;t_b-t_a)$$

$$= \int_0^\infty \frac{\Gamma(n/2)}{2\pi^{n/2}}d_l\, d^{l}_{00}(g_a^{-1}g_b)K_\rho(r_b,r_a,t_b-t_a)d\rho, \qquad (6.32)$$

with

$$K_\rho(r_b,r_a,t_b-t_a)$$

$$= (r_br_a)^{(1-n)/2}\lim_{N\to\infty}\int\prod_{j=1}^{N}\exp\left\{\frac{i}{\hbar}S^\rho_j\right\}$$

$$\times\prod_{j=1}^{N}\left(\frac{mi}{2\pi\hbar\epsilon}\right)^{1/2}\prod_{j=1}^{N-1}dr_j, \qquad (6.33)$$

$$S^\rho_j = -(m/2\epsilon)\Delta r_j^2 - [(\rho^2+\tfrac{1}{4})/2mr_jr_{j-1}]\hbar^2\epsilon.$$

The propagator on a space of constant negative curvature: For $r \equiv 1$ we set[10]

$$\sqrt{\frac{mi}{2\pi\hbar\epsilon}}\exp\left\{\frac{-im\Delta r_j^2}{2\hbar\epsilon}\right\} = \delta(r_j - r_{j-1}). \qquad (6.34)$$

The radial path integral can be performed immediately. The final integration over the end position r_b yields the propagator on a space of constant negative curvature:

$$K(\mathbf{e}_b,\mathbf{e}_a;t_b-t_a)$$
$$=\int_0^\infty d\rho \exp\left\{-\frac{i}{\hbar}E_\rho(t_b-t_a)\right\}$$
$$\times\sum_{k=0}^{\infty} Z_{\rho k}(\phi_b^{(n-1)})Z^*_{\rho k}(\phi_a^{(n-1)})$$
$$\times\frac{\Gamma((n-1)/2)}{2\pi^{(n-1)/2}}\sum_M Y_{kM}(\mathbf{e}_b)Y^*_{kM}(\mathbf{e}_a), \quad (6.35)$$

where

$$Z_{\rho k}(\phi)=\frac{\Gamma((n-2)/2+k+i\rho)}{\Gamma(i\rho)}$$
$$\times\sinh^{(3-n)/2}\phi\, P^{(3-n)/2-k}_{-1/2+i\rho}(\cosh\phi). \quad (6.36)$$

The $Y_{kM}(\mathbf{e})$ are the hyperspherical harmonics of the $(n-1)$-dimensional compact subspace [see Eq. (5.25)]. The $Z_{\rho k}(\phi)$, already discussed by Bander and Itzykson,[38] obey the orthogonality relation

$$\int_0^\infty Z_{\rho k}(\phi)Z^*_{\rho' k}(\phi)d\phi=\delta(\rho-\rho'). \quad (6.37)$$

Finally we remark that the energy spectrum is continuous,

$$E_\rho=(\rho^2+\tfrac{1}{4})(\hbar^2/2m). \quad (6.38)$$

It is, up to the additive constant, identical with that of a free particle having the momentum $p=\hbar\rho$. Therefore the above treatment may be a useful tool for solving scattering problems via path integration. The constant energy shift $\hbar^2/8m$ has also recently been obtained by Balazs and Voros.[26]

The path integral on the timelike hyperboloid $\mathscr{H}_{+1}$ can be performed similarly and has been done in Ref. 8.

B. Path integration over the SU(1,1) manifold

As a last application we consider the Feynman propagator on the group manifold of SU(1,1). The unitary irreducible representations of SU(1,1) have been constructed by Bargmann[39] for the first time. In recent years the group SU(1,1) has attracted much attention in the group theoretical approach to scattering theory.[40] In path integral formalism there exists much interest on SU(1,1) symmetries.[35] A first explicit path integral has been performed by the authors,[8,13] where the SU(1,1) manifold has been realized on the upper sheet ($x^1>0$) of a timelike hyperboloid $\mathscr{H}_{+1}$. In this section we consider quantum mechanics in $E_{2,2}$ with metric (4.8).

The spinor representation is in analogy to SU(2) parametrized in the following way:

$$g(\phi,\theta,\psi)=\begin{pmatrix} e^{i\varphi/2} & 0\\ 0 & e^{-i\varphi/2}\end{pmatrix}$$
$$\times\begin{pmatrix}\cosh\theta/2 & \sinh\theta/2\\ \sinh\theta/2 & \cosh\theta/2\end{pmatrix}\begin{pmatrix} e^{i\psi/2} & 0\\ 0 & e^{-i\psi/2}\end{pmatrix},$$
$$0\leqslant\varphi<2\pi,\quad 0\leqslant\theta<\infty,\quad 0\leqslant\psi<4\pi. \quad (6.39)$$

The associate invariant measure is

$$dg=(1/16\pi^2)\sinh\theta\, d\theta\, d\varphi\, d\psi. \quad (6.40)$$

Comparison of Eq. (6.39) with (4.11) yields the following explicit identification of the parameters with coordinates in $E_{2,2}$:

$$\begin{aligned} x^1&=r\sinh(\theta/2)\sin((\psi-\varphi)/2),\\ x^2&=r\sinh(\theta/2)\cos((\psi-\varphi)/2),\\ x^3&=r\cosh(\theta/2)\sin((\psi+\varphi)/2),\\ x^4&=r\cosh(\theta/2)\cos((\psi+\varphi)/2),\end{aligned}\qquad \begin{aligned}&0\leqslant\varphi<2\pi,\\ &0\leqslant\theta<\infty,\\ &0\leqslant\psi<4\pi.\end{aligned} \quad (6.41)$$

The Feynman ansatz is then

$$K(\mathbf{r}_b,\mathbf{r}_a;t_b-t_a)$$
$$=\lim_{N\to\infty}\int\prod_{j=1}^{N}\exp\left\{\frac{i}{\hbar}S_j\right\}\prod_{j=1}^{N}\left(\frac{m}{2\pi i\hbar\epsilon}\right)\left(\frac{im}{2\pi\hbar\epsilon}\right)$$
$$\times\prod_{j=1}^{N-1} r_j^3\, dr_j\, 2\pi^2\, dg_j, \quad (6.42)$$

$$S_j=\frac{m}{2\epsilon}\Delta r_j^2+\frac{m}{\epsilon}r_j r_{j-1}\left[1-\frac{1}{2}\mathrm{Tr}(\hat{g}_j)\right],$$

where we have made use of the results of Sec. IV A.

As is well known, the unitary irreducible representations $D^{l,\sigma}(g)$ in the Hilbert space may be divided into two fundamental and one supplementary series. The fundamental ones are (Bargmann's notation is $k=l+1$)

$$\text{continuous series: } l=-\frac{1}{2}+i\rho\begin{cases}\rho\geqslant0, & m=0,\pm1,\pm2,\ldots, \quad \text{for } \sigma=0,\\ \rho>0, & m=\pm\frac{1}{2},\pm\frac{3}{2},\ldots, \quad \text{for } \sigma=\frac{1}{2},\end{cases} \quad (6.43)$$

$$\text{discrete series: } l=-\frac{1}{2},0,\frac{1}{2},1,\ldots\begin{cases} m=l+1,l+2,\ldots, & \text{for } \sigma=+,\\ m=-l-1,-l-2,\ldots, & \text{for } \sigma=-.\end{cases} \quad (6.44)$$

The matrix elements are given by the multiplier representation

$$d^{l,\sigma}_{mn}(g)=e^{-im\varphi}d^{l,\sigma}_{mn}(\theta)e^{-in\psi}. \quad (6.45)$$

The functions $d^{l,\sigma}_{mn}(\theta)$ are called Bargmann functions and may be viewed as an analytical continuation of the Wigner polynomials $d^J_{mn}(\theta)\in$SU(2). Explicitly they are given by hypergeometric functions for $m\geqslant n$:

$$d^{l,+}_{mn}(\theta)=\frac{1}{(m-n)!}\left[\frac{\Gamma(1+m+l)\Gamma(m-l)}{\Gamma(1+n+l)\Gamma(n-l)}\right]^{1/2}\cosh^{-m-n}\frac{\theta}{2}\sinh^{m-n}\frac{\theta}{2}$$
$$\times{}_2F_1(1-n+l,-n-l;1+m-n;-\sinh^2(\theta/2)), \quad (6.46)$$

$$d^{l,-}_{mn}(\theta)=\frac{1}{(m-n)!}\left[\frac{\Gamma(1-n+l)\Gamma(-n-l)}{\Gamma(1-m+l)\Gamma(-m-l)}\right]^{1/2}\cosh^{m+n}\frac{\theta}{2}\sinh^{m-n}\frac{\theta}{2}$$
$$\times{}_2F_1(1+m+l,m-l;1+m-n;-\sinh^2(\theta/2)). \tag{6.47}$$

The functions with $n<m$ may be obtained via the relation $d^{l,\sigma}_{mn}(\theta)=(-1)^{m-n}d^{l,\sigma}_{nm}(\theta)$. For the continuous series one finds $d^{-1/2+i\rho,\sigma}_{mn}(\theta)$ by analytical continuation of Eq. (6.46) or (6.47) in $l\to-\frac{1}{2}+i\rho$. Note that for $m=n$ we have $d^{l,+}_{mm}(\theta)=d^{l,-}_{mm}(\theta)$.

According to a theorem of Bargmann,[39] the Hilbert space of square integrable functions on SU(1,1) is spanned by the fundamental continuous series and the discrete series with $l\geqslant 0$. The representations $D^{-1/2,\pm}(g)$ are excluded.

From the orthogonality[41]

$$\int_{SU(1,1)} d^{l',\sigma}_{m'n'}(g)d^{l,\sigma*}_{mn}(g)dg=\begin{cases}\dfrac{\delta_{ll'}}{2l+1}\delta_{mm'}\delta_{nn'}, & \text{for } \sigma=(+,-),\\[2ex] \dfrac{\delta(\rho-\rho')}{2\rho\tanh\pi(\rho+i\sigma)}\delta_{mm'}\delta_{nn'}, & \text{for } \sigma=(0,\frac{1}{2}),\end{cases} \tag{6.48}$$

follows the explicit Fourier decomposition

$$f(g)=\sum_\sigma\left\{\left[\sum_{2l=0}^{\infty}(2l+1)+\int_0^\infty d\rho\, 2\rho\tanh\pi(\rho+i\sigma)\right]\times\sum_{mn}\hat f_{mn}(l)d^{l,\sigma}_{nm}(g)\right\}, \tag{6.49}$$

$$\hat f_{mn}(l)=\int_{SU(1,1)} f(g)d^{l,\sigma*}_{nm}(g)dg. \tag{6.50}$$

Let $\hat\varphi_j,\hat\theta_j,\hat\psi_j$ be the parameters of $\hat g_j=g^{-1}_{j-1}g_j$, then the trace in Eq. (6.42) is given by $\frac{1}{2}\operatorname{Tr}(\hat g_j)=\cosh(\hat\theta_j/2)\cos((\hat\varphi_j+\hat\psi_j)/2)$. Therefore we have to consider the expansion of the term $\exp\{-iz\cosh(\theta/2)\cos((\varphi+\psi)/2)\}$ with $z=mr_jr_{j-1}/\hbar\epsilon$. The calculation, given in the Appendix, leads to

$$\hat f_{mn}(l)=(2/\pi z)\{K_{2l+1}(ze^{i\pi/2})+(-1)^{2m}K_{2l+1}(ze^{-i\pi/2})\}\delta_{mn}. \tag{6.51}$$

The complete expansion reads

$$\exp\left\{-\frac{iz}{2}\operatorname{Tr}(g)\right\}=\sum_\sigma\left[\sum_{2l=0}^{\infty}(2l+1)+\int_0^\infty d\rho\,2\rho\tanh\pi(\rho+i\sigma)\right]\times\frac{2}{\pi z}[K_{2l+1}(iz)+(-1)^{2m}K_{2l+1}(-iz)]\chi^{l,\sigma}(g). \tag{6.52}$$

For the path integration we do need only the asymptotic form for large $|z|$ of the expression

$$F^\sigma_l(z)=(2/\pi z)[K_{2l+1}(iz)+(-1)^{2m}K_{2l+1}(-iz)]. \tag{6.53}$$

For this we have to distinguish between the discrete and continuous case.

As the continuous series is a consequence of the noncompact nature of $E_{2,2}$ we associate this series with the integration over the noncompact coordinates (x^1,x^2), where the mass has to be regularized by a negative imaginary part ($\Rightarrow \operatorname{Im} z<0$). If we look at the asymptotic behavior of $K_\nu(iz)$,

$$K_\nu(iz)=\sqrt{\frac{\pi}{2iz}}\exp\left\{-iz+\frac{\nu^2-\frac{1}{4}}{2iz}+O\left(\frac{1}{z^2}\right)\right\}, \tag{6.54}$$

we realize, that in Eq. (6.53) the first term is increasing exponentially for $|z|\to\infty$ with $\operatorname{Im} z<0$ and the second one is damping out. Therefore we may drop the last term for continuous l [a similar argument has been used for the asymptotic form (5.22) in Ref. 16.]:

$$F^{(0,1/2)}_l(z)\approx(2/\pi z)K_{i\rho}(iz),\quad |z| \text{ large}. \tag{6.55}$$

The discrete series, however, may be associated with the compact subspace (x^3,x^4) and the regularization requires a positive imaginary part of the mass ($\Rightarrow \operatorname{Im} z>0$). Using the identity[32]

$$K_\nu(iz)=e^{-i\pi\nu}K_\nu(-iz)-i\pi I_\nu(-iz) \tag{6.56}$$

we find

$$F^\sigma_l(z)=(2/\pi z)\{(e^{-2\pi im}-e^{-2\pi il})\times K_{2l+1}(-iz)-i\pi I_{2l+1}(-iz)\}. \tag{6.57}$$

In the discrete case m and l are both integer or half-integer and therefore

$$F^{(+,-)}_l(z)=(2/iz)I_{2l+1}(-iz). \tag{6.58}$$

For $\operatorname{Im} z>0$ the asymptotic formula (5.22) is applicable. Explicitly we have in both cases

$$F^\sigma_l(z)=\frac{1}{2\pi^2}\frac{2\pi}{iz}\left(\frac{2\pi i}{z}\right)^{1/2}\times\exp\left\{-iz-i\frac{(2l+1)^2-\frac{1}{4}}{2z}+O\left(\frac{1}{z^2}\right)\right\}, \tag{6.59}$$

where $\operatorname{Im} z>0$ for $\sigma=(+,-)$ and $\operatorname{Im} z<0$ for $\sigma=(0,\frac{1}{2})$.

The expansion (6.52) may be now applied to the short time propagator. Using the orthogonality (6.48) the angular integration can be performed and we find

$$K(\mathbf{r}_b,\mathbf{r}_a;t_b-t_a)=\sum_\sigma\left\{\left[\sum_{2l=0}^{\infty}\frac{(2l+1)}{2\pi^2}\chi^{l,\sigma}(g_a^{-1}g_b)+\int_0^\infty d\rho\,\frac{2\rho\tanh\pi(\rho+i\sigma)}{2\pi^2}\chi^{-1/2+i\rho,\sigma}(g_a^{-1}g_b)\right]\times K_\sigma(r_b,r_a;t_b-t_a)\right\}, \tag{6.60}$$

with

$$K_\sigma(r_b,r_a;t_b-t_a) = (r_b r_a)^{-3/2}\lim_{N\to\infty}\int\prod_{j=1}^{N}\exp\left\{\frac{i}{\hbar}S_j^\sigma\right\} \times\prod_{j=1}^{N}\left(\frac{m}{2\pi i\hbar\epsilon}\right)^{1/2}\prod_{j=1}^{N-1}dr_j, \quad (6.61)$$

$$S_j^\sigma=\frac{m}{2\epsilon}\Delta r_j^2-\left[(2l+1)^2-\frac{1}{4}\right]\frac{\hbar^2\epsilon}{2mr_jr_{j-1}}.$$

The above SU(1,1) propagator has been recently applied to the one-dimensional modified Pöschl–Teller potential

$$V(x)=\frac{\hbar^2a^2}{2m}\left(\frac{\kappa^2-\frac14}{\sinh^2 ax}-\frac{\lambda^2-\frac14}{\cosh^2 ax}\right),\quad 0<x<\infty. \quad (6.62)$$

This case can be treated similarly to the ordinary Pöschl–Teller problem of Sec. V B. Namely, for $\theta=2ax$ the Feynman ansatz

$$K(x_b,x_a;t_a-t_b)=a\lim_{N\to\infty}\int\prod_{j=1}^{N}\exp\left\{\frac{i}{\hbar}S_j\right\}\prod_{j=1}^{N}\left(\frac{im}{2\pi\hbar a^2\epsilon}\right)^{1/2}\prod_{j=1}^{N-1}\frac12 d\theta_j, \quad (6.63)$$

$$S_j=\frac{m}{a^2\epsilon}\left(1-\cosh\frac{\Delta\theta_j}{2}\right)+\left[\frac{\kappa^2-\frac14}{\sinh(\theta_j/2)\sinh(\theta_{j-1}/2)}-\frac{\lambda^2-\frac14}{\cosh(\theta_j/2)\cosh(\theta_{j-1}/2)}-\frac14\right]\frac{a^2\hbar^2}{2m}\epsilon, \quad (6.64)$$

may be converted into a SU(1,1) path integral for $\kappa,\lambda\in N$. Note that we have used the time reversal trick of Ref. 13:

$$K(x_b,x_a;t_a-t_b)=(a/4)(\sinh\theta_b\sinh\theta_a)^{1/2}\exp\{-(i\hbar a^2/8m)(t_b-t_a)\} \times\int_0^{2\pi}\int_0^{4\pi}Q(\theta_b,\varphi_b,\psi_b;\theta_a,0,0;t_b-t_a)\exp\left\{i\left(\frac{\lambda+\kappa}{2}\varphi_b+\frac{\lambda-\kappa}{2}\psi_b\right)\right\}d\psi_b\,d\varphi_b. \quad (6.65)$$

With $Q(\theta_b,\varphi_b,\psi_b;\theta_a,0,0;t_b-t_a)$ the SU(1,1) symmetry of (6.62) is realized:

$$Q(\theta_b,\varphi_b,\psi_b;\theta_a,0,0;t_b-t_a)=\lim_{N\to\infty}\int\prod_{j=1}^{N}\exp\left\{\frac{i}{\hbar}\widetilde S_j\right\}\prod_{j=1}^{N}\left(\frac{m}{2\pi i\hbar a^2\epsilon}\right)^{1/2}\left(\frac{-im}{2\pi\hbar a^2\epsilon}\right)\prod_{j=1}^{N-1}2\pi^2 dg_j, \quad (6.66)$$

$$\widetilde S_j=(m/a^2\epsilon)[1-\tfrac12\operatorname{Tr}(\hat g_j)]. \quad (6.67)$$

The integration gives

$$K(x_b,x_a;t_b-t_a) =a(\sinh\theta_b\sinh\theta_a)^{1/2} \times\Bigg\{\sum_{l=\sigma}^{(\lambda-\kappa)/2-1}(2l+1)d^{\,l,\sigma}_{(\lambda+\kappa)/2,(\lambda-\kappa)/2}(\theta_b)d^{\,l,\sigma*}_{(\lambda+\kappa)/2,(\lambda-\kappa)/2}(\theta_a)\exp\left\{\frac{i}{\hbar}(2l+1)^2\frac{\hbar^2a^2}{2m}(t_b-t_a)\right\} +\int_0^\infty dp\,2p\tanh\pi(p+i\sigma)d^{\,-1/2+ip,\sigma}_{(\lambda+\kappa)/2,(\lambda-\kappa)/2}(\theta_b)d^{\,-1/2+ip,\sigma*}_{(\lambda+\kappa)/2,(\lambda-\kappa)/2}(\theta_a)\exp\left\{-\frac{i}{\hbar}\frac{(2ap)^2\hbar^2}{2m}(t_b-t_a)\right\}\Bigg\}, \quad (6.68)$$

where $\sigma=0$ ($\frac12$) for $\kappa+\lambda$ even (odd). With $k=2ap$ we find the standard form

$$K(x_b,x_a;t_b-t_a)=\sum_{l=\sigma}^{(\lambda-\kappa)/2-1}e^{-i/\hbar E_l(t_b-t_a)}\Psi_l(x_b)\Psi_l^*(x_a)+\int_0^\infty dk\,e^{-i/\hbar E_k(t_b-t_a)}\Phi_k(x_b)\Phi_k^*(x_a), \quad (6.69)$$

where the bound and scattering states are found simultaneously via path integration:

$$E_l=-(2l+1)^2\frac{\hbar^2a^2}{2m},\quad \Psi_l(x)=[a(2l+1)\sinh 2ax]^{1/2}d^{\,l,\sigma}_{(\lambda+\kappa)/2,(\lambda-\kappa)/2}(2ax),$$
$$E_k=\frac{\hbar^2k^2}{2m},\quad \Phi_k(x)=\left[\frac{k}{2a}\sinh 2ax\tanh\pi\left(\frac{k}{2a}+i\sigma\right)\right]^{1/2}d^{\,-1/2+i(k/2a),\sigma}_{(\lambda+\kappa)/2,(\lambda-\kappa)/2}(2ax). \quad (6.70)$$

The energy eigenvalues and eigenfunctions are identical with that obtained by the algebraic method.[42]

The above technique is also applicable to the Coulomb problem in a space of constant positive curvature.[14]

Path integration on SO(n,m) *in bispherical coordinates:* The path integral solution of the modified Pöschl–Teller problem may be used for the calculation of the Feynman propagator in $E_{n,m}$. Choosing the subspace T_{+1} with the parametrization

$$
\begin{aligned}
x^1 &= r\cosh\frac{\theta}{2}\sin\alpha^{(n-1)}\cdots\sin\alpha^{(1)},\\
&\vdots\\
x^n &= r\cosh\frac{\theta}{2}\cos\alpha^{(n-1)}, && 0\leqslant r,\theta<\infty,\\
& && 0\leqslant\alpha^{(1)},\beta^{(1)}<2\pi,\\
x^{n+1} &= r\sinh\frac{\theta}{2}\sin\beta^{(m-1)}\cdots\sin\beta^{(1)}, && 0\leqslant\alpha^{(i)},\beta^{(i)}<\pi \quad (i\neq 1),\\
&\vdots\\
x^{n+m} &= r\sinh\frac{\theta}{2}\cos\beta^{(m-1)},
\end{aligned}
\tag{6.71}
$$

the Jacobian is

$$
d^{m+n}\mathbf{r} = r^{m+n-1}\,dr\,\frac{1}{2}\sinh^{m-1}\frac{\theta}{2}\cosh^{n-1}\frac{\theta}{2}\,d\theta\, d^{n-1}\Omega(\alpha)d^{m-1}\Omega(\beta),
\tag{6.72}
$$

and the propagator reads

$$
K(\mathbf{r}_b,\mathbf{r}_a;t_b-t_a) = \lim_{N\to\infty}\int\prod_{j=1}^{N}\exp\left\{\frac{i}{\hbar}S_j\right\}\prod_{j=1}^{N}\left(\frac{m}{2\pi i\hbar\epsilon}\right)^{n/2}\left(\frac{mi}{2\pi\hbar\epsilon}\right)^{m/2}\prod_{j=1}^{N-1}d^{m+n}\mathbf{r}_j,
\tag{6.73}
$$

$$
\begin{aligned}
S_j &= \frac{m}{2\epsilon}\Delta r_j^2 + \frac{m}{\epsilon}r_jr_{j-1}\left(1-\cosh\frac{\Delta\theta_j}{2}\right)\\
&\quad+\frac{m}{\epsilon}r_jr_{j-1}\cosh\frac{\theta_j}{2}\cosh\frac{\theta_{j-1}}{2}\,[1-\mathbf{e}^{\alpha}_{j-1}\cdot\mathbf{e}^{\alpha}_{j}] - \frac{m}{\epsilon}r_jr_{j-1}\sinh\frac{\theta_j}{2}\sinh\frac{\theta_{j-1}}{2}\,[1-\mathbf{e}^{\beta}_{j-1}\cdot\mathbf{e}^{\beta}_{j}].
\end{aligned}
\tag{6.74}
$$

After integration over the $\alpha^{(i)}$'s and $\beta^{(i)}$'s using $SO(n,m)\supset SO(n)\times SO(m)$ we have

$$
\begin{aligned}
K(\mathbf{r}_b,\mathbf{r}_a;t_b-t_a) &= \sum_{l,\lambda=0}^{\infty}K_{l,\lambda}(r_b,\theta_b;r_a,\theta_a;t_b-t_a)\\
&\quad\times\frac{\Gamma(n/2)}{2\pi^{n/2}}\sum_N Y_{lN}(\mathbf{e}^{\alpha}_b)Y^*_{lN}(\mathbf{e}^{\alpha}_a)\frac{\Gamma(m/2)}{2\pi^{m/2}}\sum_M Y_{\lambda M}(\mathbf{e}^{\beta}_b)Y^*_{\lambda M}(\mathbf{e}^{\beta}_a),
\end{aligned}
\tag{6.75}
$$

with

$$
\begin{aligned}
K_{l,\lambda}(r_b,\theta_b;r_a,\theta_a;t_b-t_a) &= \left(r_br_a\sinh\frac{\theta_b}{2}\sinh\frac{\theta_a}{2}\right)^{(1-m)/2}\left(r_br_a\cosh\frac{\theta_b}{2}\cosh\frac{\theta_a}{2}\right)^{(1-n)/2}\\
&\quad\times\lim_{N\to\infty}\int\prod_{j=1}^{N}\exp\left\{\frac{i}{\hbar}\tilde S_j\right\}\prod_{j=1}^{N}\left(\frac{m}{2\pi i\hbar\epsilon}\right)^{1/2}\left(\frac{mi}{2\pi\hbar\epsilon}\right)^{1/2}\prod_{j=1}^{N-1}r_j\,dr_j\,\frac{1}{2}\,d\theta_j,
\end{aligned}
\tag{6.76}
$$

$$
\tilde S_j = \frac{m}{2\epsilon}\Delta r_j^2 + \frac{m}{\epsilon}r_jr_{j-1}\left(1-\cosh\frac{\Delta\theta_j}{2}\right) + \left[\frac{\mu^2-\frac14}{\sinh(\theta_j/2)\sinh(\theta_{j-1}/2)} - \frac{\nu^2-\frac14}{\cosh(\theta_j/2)\cosh(\theta_{j-1}/2)}\right]\frac{\hbar^2\epsilon}{2mr_jr_{j-1}},
\tag{6.77}
$$

where we have defined $\mu=\lambda+(m-2)/2$ and $\nu=l+(n-2)/2$. The remaining θ integration may be transformed into an SU(1,1) path integral. Indeed,it is formally identical with the modified Pöschl–Teller problem. Using this result leads to

$$
\begin{aligned}
K_{l,\lambda}(r_b,\theta_b;r_a,\theta_a;t_b-t_a) &= 2\left(r_br_a\sinh\frac{\theta_b}{2}\sinh\frac{\theta_a}{2}\right)^{(2-m)/2}\left(r_br_a\cosh\frac{\theta_b}{2}\cosh\frac{\theta_a}{2}\right)^{(2-n)/2}\\
&\quad\times\left[\sum_{l=\sigma}^{(\nu-\mu)/2-1}(2J+1)+\int_0^{\infty}d\rho\,2\rho\tanh\pi(\rho+i\sigma)\right]K_l(r_b,r_a;t_b-t_a)\\
&\quad\times d^{l,\sigma}_{(\nu+\mu)/2,(\nu-\mu)/2}(\theta_b)\,d^{l,\sigma*}_{(\nu+\mu)/2,(\nu-\mu)/2}(\theta_b),
\end{aligned}
\tag{6.78}
$$

where $K_l(r_b,r_a;t_b-t_a)$ is given by Eq. (5.37) for $J=l$ and $\sigma=0$ or $\frac12$ for $(\nu-\mu)$ even or odd, respectively. For $r\equiv1$ we have the spectrum $K_l(1,1;t_b-t_a)=\exp\{-(i/\hbar)E_l(t_b-t_a)\}$ with

$$
E_l = \begin{cases}[(2l+1)^2-\frac14](\hbar^2/2m), & \text{for } l \text{ discrete},\\ -(\rho^2+\frac14)(\hbar^2/2m), & \text{for } l=-\frac12+i\rho.\end{cases}
\tag{6.79}
$$

VII. DISCUSSION AND OUTLOOK

In the present paper we have discussed the path integral on compact and noncompact rotation groups. The group manifold has been embedded into Euclidean and pseudo-Euclidean spaces, respectively. For this we had to generalize the usual path integral formalism, where the construction is very much similar to that of Feynman. Especially the regularizing scheme had to be modified in order to get well-defined Feynman integrals. Restricting the discussion to a connected subspace of the pseudo-Euclidean space the introduction of polar coordinates leads to a separation into a radial and angular part.

Application of group theory enables us to perform the angular integration, where group theory is introduced through identification of coordinates with group parameters. Writing the short time propagator as a function of group elements, the Fourier analysis on the group leads to an expansion of the propagator in unitary irreducible representations. We have found two methods. For $\dim G = \dim \mathscr{H}_\alpha$ the short time action may be written in terms of the character of the fundamental representation, $\chi(g^{(f)}) = \mathrm{Tr}\, g^{(f)}$. Note that the short time action is formally identical with the Wilson action in lattice gauge theories. The character expansion, which has been already used extensively in lattice gauge theories, is applicable. The angular integration reduces to an application of the orthogonality relation of group characters. The only simple Lie groups that may be treated in this way are SU(2) and SU(1,1). In the general case $\dim G \geqslant \dim \mathscr{H}_\alpha$ and the short time action is, by construction, invariant under transformations of the subgroup H as $\mathscr{H}_\alpha = G/H$. Here the expansion in zonal spherical functions is a proper treatment and the application of their orthogonality relation enables us to perform the angular integrals. In both cases the remaining radial path integral is expressed in terms of modified Bessel functions of the first and third kind for compact and noncompact groups, respectively.

The formalism has been applied to the physically most important groups. For the compact groups SO(n) and SU(2) we have recovered known expansion formulas, which have found many applications in path integration in the recent years. For noncompact groups such an expansion has been applied in path integration only by the authors.[13] Here we have chosen the n-dimensional Lorentz group SO($n-1$,1) and the group SU(1,1). The SO($n-1$,1) propagator is found to have the continuous spectrum of a free particle and therefore may become an important tool in scattering theory via path integration [e.g., Rutherford scattering has a SO(3,1) symmetry]. The spectrum generating property of the SU(1,1) algebra, which has numerous applications in group theory, has been used in path integration, too. With the SU(1,1) propagator the bound and scattering states of various problems (here the modified Pöschl–Teller potential has been taken) may be found simultaneously.

Besides nonrelativistic quantum theory the proposed expansion methods may also be very useful in quantum field theories. For pure Yang–Mills lattice gauge theories the character expansion has already been used for a long time. It would be interesting to know whether the expansion in zonal spherical functions still works in theories with matter fields like the symmetry breaking Higgs field. Another area of applications is the path integral formalism of statistical physics. Here the partition function is given as a functional integral over the Boltzmann factor $\exp\{-\beta H\}$. Especially for scalar theories the expansion in zonal spherical functions seems to be successful. How far both techniques may be applied in field theories of elementary particle and solid state physics is under present investigation by the authors.[43]

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APPENDIX: DERIVATION OF THE SU(1,1) FOURIER COEFFICIENT

The formula (6.50) of the Fourier coefficient reads, for $f(g) = \exp\{-(iz/2)\mathrm{Tr}(g)\}$,

$$\hat{f}_{mn}(l) = \frac{1}{16\pi^2}\int_0^{4\pi}\int_0^{2\pi}\int_0^{\infty} \exp\left\{-iz\cosh\left(\frac{\theta}{2}\right)\cos\frac{\varphi+\psi}{2}\right\} \times e^{-im\varphi}e^{-in\psi}d^{\,l,\sigma}_{mn}(\theta)\sinh\theta\, d\theta\, d\varphi\, d\psi. \tag{A1}$$

Using the generating function of Bessel functions,

$$\exp\left\{-iz\cosh\left(\frac{\theta}{2}\right)\cos\frac{\varphi+\psi}{2}\right\} = \sum_{2p=-\infty}^{\infty} e^{ip(\varphi+\psi)}e^{-i\pi p}J_{2p}\left(z\cosh\left(\frac{\theta}{2}\right)\right), \tag{A2}$$

the integrals over φ and ψ may be performed and yield

$$\hat{f}_{mn}(l) = \frac{\delta_{mn}}{2}e^{-i\pi m}\int_0^{\infty} J_{2m}\left(z\cosh\left(\frac{\theta}{2}\right)\right)\cosh^{2m}(\theta/2) \times {}_2F_1(l+m+1,m-l;1;-\sinh^2(\theta/2)) \times \sinh\theta\, d\theta, \tag{A3}$$

where we have used the explicit form (6.47) of the Bargmann functions. Writing the Bessel function in terms of the Meijer G function we find with $x = \cosh^2(\theta/2)$:

$$\hat{f}_{mn}(l) = \delta_{mn}e^{-i\pi m}\int_1^{\infty} x^m G^{10}_{02}\left(\frac{xz^2}{4}\middle| m,-m\right) \times {}_2F_1(l+m+1,m-l;1;1-x^2)\, dx. \tag{A4}$$

This integral is a special case of the integral formula #7.831 in Ref. 44. The second set of integrability conditions gives for $\mathrm{Re}\, l \geqslant -\frac{1}{2}$,

$$\hat{f}_{mn}(l) = \delta_{mn}e^{-i\pi m}G^{30}_{24}\left(\frac{z^2}{4}\middle|\begin{matrix} m,-m \\ l,-l-1,m,-m\end{matrix}\right). \tag{A5}$$

Using some properties[45] of the G function the order may be reduced in three steps and the final G functions can be identified with modified Bessel functions of the third kind:

$$\begin{aligned}\hat{f}_{mn}(l) &= \delta_{mn}e^{-i\pi m}G^{20}_{13}\left(\frac{z^2}{4}\middle|\begin{matrix}-m \\ l,-l-1,-m\end{matrix}\right)\\ &= \frac{\delta_{mn}}{2\pi i}e^{-i\pi m}\left\{e^{-i\pi m}G^{21}_{13}\left(\frac{z^2}{4}e^{-i\pi}\middle|\begin{matrix}-m \\ l,-l-1,-m\end{matrix}\right) - e^{i\pi m}G^{21}_{13}\left(\frac{z^2}{4}e^{i\pi}\middle|\begin{matrix}-m \\ l,-l-1,-m\end{matrix}\right)\right\}\\ &= \frac{\delta_{mn}}{2\pi i}\left\{e^{-2\pi i m}G^{20}_{02}\left(\frac{z^2}{4}e^{-i\pi}\middle| l,-l-1\right) - G^{20}_{02}\left(\frac{z^2}{4}e^{i\pi}\middle| l,-l-1\right)\right\}\\ &= \frac{2}{\pi z}\{K_{2l+1}(ze^{i\pi/2}) + e^{-2\pi i m}K_{2l+1}(ze^{-i\pi/2})\}\delta_{mn}.\end{aligned} \tag{A6}$$

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